STABILITY FOR A FUNCTIONAL DIFFERENTIAL EQUATION IN HILBERT SPACE

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ABSTRACT
The stability for the functional differential equation: du/dt = Au(t)+bu(t)+(a*Au)(t) is studied, where A is the infinitesimal generator of a linear dynamical system in Hilbert space and the convolution term contains a square integrable real function a. Sufficient conditions for the asymptotic stability of the solution u are obtained. The results are applied to a retarded partial integrodifferential equation.

Key words: linear dynamical system, embedding.

RESUMEN
La estabilidad de la ecuación diferencial funcional du/dt = Au(t)+bu(t)+(a*Au)(t) es estudiada, donde A es el generador infinitesimal del sistema dinámico lineal en un espacio de Hilbert y el término de convolución contiene una función cuadrada real integrable a. Condiciones suficientes para la estabilidad asintótica de la solución u son obtenidas. Los resultados son aplicados a una ecuación integrodiferencial con retardo.

Palabras clave: sistema dinámico lineal,

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1. INTRODUCTION

We consider the stability properties for the functional differential equation (FDE) of the form:

$$
\begin{aligned}
&u'(t) = Au(t) + bu(t) + \int_{-h}^{0} a(r)Au(t+r)dr \qquad t > 0 \\
&u(0) = \phi^0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1.1)\\
&u(r) = \phi^1(r) \qquad r \in [-h,0)
\end{aligned}
$$

where A is the infinitesimal generator of an analytic semigroup of linear operators S(t) on a Hilbert space X, b is a real number, a(.) is measurable square integrable real function and the initial value $\phi = (\phi^0,\phi^1)$ belongs to the product space $F\times L^2(-h,0;D(A)) = Z$ with F denoting a suitable intermediate space between D(A) and X. We note that the concept of a linear dynamical system $\{S(t);\ t \geq 0\}$ is equivalent to that of a C_0-semigroup; see e.g. [Walker (1980)]. In this paper the notation of semigroups will be used.

For any $\phi \in Z$ there exists a solution u = u(t) which satisfies equation (1.1) for a.e. $t \in [0,T]$; see e.g. [1]. Thus we can define the solution semigroup T(t) in the product space Z. Consequently we can consider the stability of the solution u(t) of (1.1) by studying the asymptotic behaviour of the solution semigroup T(t) in Z. By the strong continuity of the semigroup T(t) (for the proof see e.g. [1]) there exist real constants M and ω such that

$$\|T(t)\| \leq Me^{\omega t} \qquad t \geq 0 \qquad (1.2)$$

Let Λ be the infinitesimal generator of the solution semigroup T(t) in Z and $\sigma(\Lambda)$ its spectrum. It is known, when the semigroup T(t) is norm continuous (i.e. it is continuous in the uniform operator topology) for $t \geq t_0 > 0$, then the following relation holds:

$$\sup\{\mathrm{Re}\,\lambda:\ \lambda \in \sigma(\Lambda)\} = \inf\{\omega \in R;\ \|T(t)\| \leq Me^{\omega t} \text{ for some } M > 0\} \qquad (1.3)$$

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and the study of the asymptotic behaviour of the semigroup T(t) can be made through the location of the spectrum of Λ.

There are various conditions on the semigroup such that T(t) is norm continuous for $t \geq t_0$ and (1.3) holds. For instance, for the case when b = 0, Di Blasio, Kunisch and Sinestrari in [1985] have proved: when the weight function $a(\cdot)$ appearing in the distributed delay term of (1.1) belongs to $W^{1,2}(-h,0;R)$, then the associated solution semigroup T(t) is a differentiable semigroup for t > h and thus (1.3) is fulfilled. In Jeong (1991) has shown: when $a(\cdot)$ is Hoelder continuous, then the solution semigroup T(t) is norm Hoelder continuous for t > 3h.

The objective of this paper is to establish norm continuity for $t \geq t_0$ also for the case, where $b \neq 0$ and the weight function $a(\cdot)$ is not necessarily continuous. In section 3 we prove the following result: when a(.) is a measurable square integrable real function ($a \in L^2(-h,0;R)$), then the solution semigroup is norm continuous for t > h. In section 4 we use the norm continuity of the solution semigroup and consider the stability of the solution $u(\cdot)$ of the equation (1.1). By known results this can be done by studying the characteristic equation of a given functional differential equation; see e.g. Hale [1977] and Travis and Webb [1974]. We will study solutions of the characteristic equation of (1.1) and prove stability of u(.) for the case where the infinitesimal generator A has real point spectrum such that $\sigma(A) = \sigma_p(A) \subseteq (-\infty,-\alpha_0]$ for some $\alpha_0 > 0$. At the end an example with initial boundary value problem for a retarded partial integrodifferential equation is considered.

Notation: Throughout X is a Hilbert space with norm $\|\cdot\|$, R and C are the sets of real respectively complex numbers and L(X) is the space of bounded linear operators in X. For a closed linear injective operator A in X its domain D(A) is regarded as a Banach space with the norm $\|x\|_{D(A)} = \|Ax\|$. $C(t_0,T;Y)$ is the space of continuous functions from an interval $[t_0,T]$ to a Banach space Y and $L^2(t_0,T;Y)$ denotes the space of measurable square integrable functions from $[t_0,T]$ to Y. As usual $W^{1,2}(t_0,T;Y)$ denotes the space of absolutely continuous functions on $[t_0,T]$ such that their derivative belongs to $L^2(t_0,T;Y)$.

2. SOLUTION SEMIGROUP

Let A be the infinitesimal generator of an analytic semigroup $\{S(t);\ t \geq 0\}$ on X with 0 in the resolvent set, such that:

$$\|S(t)\|_{L(X)} \leq M_0 \qquad t \geq 0 \tag{2.1}$$

$$\|tAS(t)\|_{L(X)} \leq M_1 \quad t > 0, \tag{2.2}$$

for some constants M_0 and M_1.

In order to obtain the existence and stability results on the solution of the functional differential equation (1.1) one usually studies the equivalent integral equation:

$$u(t) = S(t)\phi^0 + b\int_0^t S(t-s)u(s)ds + \int_0^t S(t-s)\int_{-h}^0 a(r)Au(s+r)drds \quad t \geq 0 \tag{2.3}$$

For $(\phi^0,\phi^1) \in X \times L^2(-h,0;X)$ the initial value problem is not well posed. The appropriate initial data is an element of the product space $Z = F \times L^2(-h,0;D(A))$, where F is the Lions real interpolation space between D(A) and X given by:

$$F = \left\{ x \in X;\ \int_0^\infty \|AS(t)x\|^2 dt < \infty \right\} \qquad \text{and norm} \tag{2.4}$$

$$\|x\|_F = \left(\|x\|^2 + \int_0^\infty \|AS(t)x\|^2 dt \right)^{1/2}$$

For the details see e.g. [1]. Moreover, there is a continuous embedding:

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;X) \to C(0,T;F) \tag{2.5}$$

so that there is aconstant c_0 such that

$$\|u\|_{C(0,T;F)} \le c_0 \|u\|_{L^2(0,T;D(A)) \cap W^{1,2}(0,T;X)} \tag{2.6}$$

for each $u \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;X)$; see e.g. [1], [7]. Throughout the paper we will assume $a(\cdot)$ to be a measurable square integrable function:

Assumption 1: $a(\cdot) \in L^2(-h,0;R)$ (2.7)

By assuming (2.7) the existence of solution of FDE (1.1) has been proved in [1]. More precisely: the following result was obtained: for an arbitrary initial value $\phi = (\phi^0, \phi^1) \in Z$ there is a unique solution $u = u(t)$ of FDE (1.1) on the interval $[-h,T]$, $T > 0$ such that $u \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;X)$,

$$\|u\|_{L^2(0,T;D(A)) \cap W^{1,2}(0,T;X)} \le c_1 \|\phi\|_Z \tag{2.8}$$

and the constant c_1 depends on T, M_0, M_1, $\|a\|$, b. Thus the solution semigroup $T(t)$ can be defined in the product space $Z = F \times L^2(-h,0;D(A))$ in the following way:

$$T(t)\phi = (u(t), u_t(\cdot)) \qquad t \ge 0 \tag{2.9}$$

for every $\phi \in Z$ and u_t given by $u_t(s) = u(t+s)$, $s \in [-h,0]$. As noted, this semigroup is strongly continuous in Z; see e.g. [1]. In the next section we will show that this semigroup is in fact norm continuous for $t>h$.

3. NORM CONTINUITY

In this section we consider the continuity of the solution semigroup $T(t)$ for $t > h$ in the uniform operator topology. Actually we can prove that the solution semigroup of FDE (1.1) is norm continuous uniformly in t on any closed interval $[t_0,T]$ for $t_0 > h$. More precisely:

PROPOSITION 1. Let $\{T(t);\ t \ge 0\}$ be the solution semigroup of the functional differential equation (1.1) and suppose that $0 < h < t_0 \le \tau \le \tau' \le T$, with t_0 and T fixed. Then for every $\varepsilon > 0$ there is a $\Delta > 0$ such that

$$\|T(\tau')\phi - T(\tau)\phi\|_Z < \varepsilon \|\phi\|_Z \tag{3.1}$$

holds for every $\phi \in Z$ whenever $(\tau' - \tau) < \Delta$. The solution semigroup $\{T(t);\ t \ge 0\}$ of the functional differential equation (1.1) is continuous in the uniform operator topology at every $t > h$.

Proof. Let $u \in L^2(-h,T;D(A))$ be the solution of equation (1.1) and $0 < h < \tau_0 \le \tau \le \tau' \le T$. We have to show that

$$\|T(\tau')\phi - T(\tau)\phi\|_Z = \|u(\tau') - u(\tau)\|_F + \|u_{\tau'} - u_\tau\|_{L^2(-h,0;D(A))} \le \frac{\varepsilon}{2}\|\phi\|_Z + \frac{\varepsilon}{2}\|\phi\|_Z = \varepsilon\|\phi\| \tag{3.2}$$

holds for every $\phi \in Z$, whenever $(\tau' - \tau) \le \Delta$.

First let us consider the L^2 norm of the second term. By definition we have:

$$\|u_{\tau'} - u_\tau\|^2_{L^2(-h,0;D(A))} = \int_{-h}^{0} \|Au(\tau'+r) - Au(\tau+r)\|^2 dr = \tag{3.3}$$

Denoting $\tau' - \tau = \Delta$, $t = \tau + r$ and $t_0 = \tau_0 - h$ we get:

$$= \int_{\tau-h}^{\tau} \|Au(t+\Delta) - Au(t)\|^2 dt \le$$

$$\le \int_{t_0}^{T-\Delta} \|Au(t+\Delta) - Au(t)\|^2 dt = \|u(t+\Delta - u(t)\|^2_{L^2(t_0, T-\Delta; D(A))} \tag{3.4}$$

In order to estimate the last integral (3.4) we first rewrite the integral equation (2.3) as:

$$\begin{aligned} u(t) &= S(t)\phi^0 + b\int_0^t S(t-s)u(s)ds + \int_0^t S(t-s)\int_{-h}^0 a(r)Au(s+r)drds = \\ &= S(t)\phi^0 + bw(t) + v(t) \end{aligned} \tag{3.5}$$

where

$$w(t) = \int_0^t S(t-s)u(s)ds \quad \text{and}$$

$$v(t) = \int_0^t S(t-s) \int_{-h}^0 a(r)Au(s+r)drds$$

By (3.4) we need the estimate of the difference of u(t) which can be written in the following form:

$$u(t+\Delta) - u(t) = \big(S(t+\Delta) - S(t)\big)\phi^0 + b\big(w(t+\Delta) - w(t)\big) + \big(v(t+\Delta) - v(t)\big) \tag{3.6}$$

Let us consider the terms on the righthand side of the equation. For the differences of $S(t)\phi^0$ and v(t) the L^2-estimates were given in [8], where it was shown that for arbitrary $t_0 > 0$ and $\varepsilon_1 > 0$ we have:

$$\left\| \big(S(t+\Delta) - S(t)\big)\phi^0 \right\|_{L^2(t_0, T; D(A))} \le \varepsilon_1 \|\phi\|_Z$$

and (3.7)

$$\|v(t+\Delta) - v(t)\|_{L^2(0, T-\Delta; D(A))} \le \varepsilon_1 \|\phi\|_Z$$

whenever Δ is sufficiently small. Thus we only need to find the estimate for the difference of w(t). Let us rewrite this term as follows:

$$\begin{aligned} w_0(t) = w(t+\Delta) - w(t) &= \int_0^{t+\Delta} S(t+\Delta-s)u(s)ds - \int_0^t S(s)u(s)ds = \\ &= \int_0^{t+\Delta} S(s+\Delta-s)u(s)ds + \int_0^t [S(t+\Delta-s) - S(t-s)]u(s)\,ds = w_1(t) + w_2(t) \end{aligned} \tag{3.8}$$

By assumption A is a closed operator, $u \in L^2(-h, T; D(A))$ and S(t) commutes with A. Therefore we may take A into the integrand (see e.g. [4]) and write:

$$Aw_1(t) = \int_t^{t+\Delta} S(t+\Delta-s)Au(s)ds$$

For every $t \in [0, T\text{-}\Delta]$ we have:

$$\|Aw_1(t)\| = M_0 \int_0^{t+\Delta} \|Au(s)\| ds$$

Moreover, by the Hoelder inequality it follows

$$\|Aw_1(t)\| \le M_0 \sqrt{\Delta} \|u\|_{L^2(0,T;D(A))} \tag{3.9}$$

In order to obtain the estimate of $w_2(t)$ we note that from (2.1) and (2.2) it follows:

$$\|S(t+\Delta) - S(t)\|_{L(X)} \le 2M_0 \qquad \text{and}$$

$$\|S(t+\Delta) - S(t)\|_{L(X)} \le M_1 \frac{\Delta}{t}$$

for $t \in (0, T\text{-}\Delta)$. Thus we have:

$$\|S(t+\Delta) - S(t)\|_{L(X)} \le M_3 k(t) \tag{3.10}$$

where $M_3=\max(2M_0, M_1)$ and

$$k(t) = \min\left(1, \frac{\Delta}{t}\right) \qquad t > 0 \tag{3.11}$$

Hence by the same argument as above we get:

$$\|Aw_2(t)\| = M_3 \int_0^t k(t-s) \|Au(s)\| ds$$

By the Hoelder inequality then it follows:

$$\|Aw_2(t)\| \le M_3 \left(\int_0^t (k(t-s))^2 ds \right)^{\frac{1}{2}} \|u\|_{L^2(0,T;D(A))}$$

By direct integration of k the following estimate is obtained:

$$\int_0^t (k(t-s)^2 ds) \le \int_0^{t-\Delta} + \int_{t-\Delta}^t \le 2\Delta$$

(see e.g. [9], Thm 3.1, p. 110). Thus it follows:

$$\|Aw_2(t)\| \le M_3 \sqrt{2}\sqrt{\Delta} \|u\|_{L^2(0,T-\Delta;D(A))} \tag{3.12}$$

By the estimates (3.9) and (3.12) then it follows:

$$\|Aw_0(t)\| \le M\sqrt{\Delta} \|u\|_{L^2(0,T-\Delta;D(A))}$$

for some M and each $t \in [0, T\text{-}\Delta]$. Hence by integration we get:

$$\|w_0\|_{L^2(t_0,T-\Delta;D(A))} \le M\sqrt{\Delta}\sqrt{T-t_0}\|u\|_{L^2(-h,T;D(A))} \tag{3.13}$$

By estimate (2.8) it also follows that

$$\|w_0\|_{L^2(t_0,T-\Delta;D(A))} \le \varepsilon_2\|\phi\|_Z \tag{3.14}$$

for any $\varepsilon_2 > 0$ and Δ sufficiently small.

By applying the estimates (3.7) and (3.14) to the equation (3.6) we get:

$$\|u(t+\Delta)-u(t)\|_{L^2(t_0,T-\Delta;D(A))} \le \frac{\varepsilon}{2}\|\phi\|_Z \tag{3.15}$$

for any $\varepsilon > 0$ and Δ sufficiently small. Moreover, from (3.3) it follows

$$\|u_{\tau'} - u_\tau\|_{L^2(-h,0;D(A))} \le \frac{\varepsilon}{2}\|\phi\|_Z$$

which proves the second estimate in (3.2). By using the continuous embedding (2.8), (2.9) the first estimate in (3.2) can be obtained exactly in the same way as it is shown for the case b = 0 in [8], hence we omit the details. Therefore from the estimate (3.2) we may conclude that the solution semigroup is norm continuous uniformly over any closed interval $[t_0,T]$ and $t_0 > 0$.This shows that it is norm continuous at every t > h as stated by Proposition 1.

4. ASYMPTOTIC STABILITY

As we have noted in the introduction, the norm continuity of T(t) for t>h implies, that the spectrum of the solution semigroup T(t) can be determined through the location of the spectrum of its infinitesimal generator Λ. It is known that for a functional differential equation the spectrum of Λ can be obtained by the so called characteristic equation of equation (1.1). Therefore we define for every complex number λ the operator $\Delta(\lambda)$: $D(A) \to X$:

$\Delta(\lambda)x = \lambda x - bx - Ax - m(\lambda)Ax$, where

$$m(\lambda) = 1 + \int_{-h}^{0} a(s)e^{\lambda s}ds$$

The equation $\Delta(\lambda)x = 0$ is the characteristic equation of equation (1.1). As noted, the spectrum of Λ can be determined from solutions of the characteristic equation. This was done in detail in [Di Blasio, **et al**. (1985)]. We will apply some of those results to the case where the infinitesimal generator A has only real spectrum satisfying the assumption:

Assumption 2 : $\sigma(A) = \sigma_p(A) \subseteq (-\infty, -\alpha_0]$, for some $\alpha_0 > 0$ (4.1)

In that case the following characterization of the spectrum of Λ can be obtained by the same argument as that given for the case b = 0 in Proposition 5.2 in [2]: If the function $a(\cdot)$ is such that $m(0) \neq 0$, then the spectrum of Λ satisfies the relation: $\sigma(\Lambda) \subseteq \Gamma_1 \cup \Gamma_0$, where

$$\Gamma_1 = \left\{\lambda \in C;\ m(\lambda) \neq 0 \text{ and } \frac{\lambda - b}{m(\lambda)} \in \sigma(A)\right\} \quad \text{and} \tag{4.2}$$

$$\Gamma_0 = \{\lambda \in C; \lambda \neq 0 \text{ and } m(\lambda) = 0\} \tag{4.3}$$

The condition $m(0) \neq 0$ means that 0 is not in the spectrum of Λ. For instance this occurs when $\|a\|_{L^1} < 1$.

From the characterizations (4.2) and (4.3) of the spectrum of Λ it follows that the stability properties of the solution can be given in terms of the weight function $a(\cdot)$. In that direction we can prove the following result.

PROPOSITION 2. Suppose A is the infinitesimal generator of a bounded analytic semigroup on X satisfying condition (4.1), $b \le 0$ and $a(.)$ is a weight function such that

$$\|a\|_{L^1(-h,0;R)} < 1 \tag{4.4}$$

Then the solution u of the functional differential equation (1.1) is asymptotically stable and the following estimate holds

$$\|u(t)\|_F + \|u_t\|_{L^2(-h,0;D(A))} \le Me^{-\omega t}\|\phi\|_Z, \tag{4.5}$$

for some $M > 0$, $\omega > 0$ and all $t \ge 0$.

Proof. Let $\omega_0 \in (0,\alpha_0)$ be arbitrary, so that $\varepsilon_0 = \omega_0/\alpha_0 < 1$. Furthermore let us define

$$a_0 = (1-\varepsilon_0)e^{-\omega_0 h} \tag{4.6}$$

We will show that under the assumption

$$\|a\|_{L^1(-h,0;R)} < a_0$$

it follows that $\mathrm{Re}\lambda \le -\omega_0$ for $\lambda \in \Gamma_1 \cup \Gamma_0$ and hence for every $\lambda \in \sigma(\Lambda)$.

Suppose first that $\lambda \in \Gamma_1$. Hence by (4.2) λ satisfies the equation $(\lambda\text{-}b)/m(\lambda) = -\alpha$ for some $\alpha > \alpha_0$ and $-\alpha \in \sigma(A)$. The equation can be rewritten as

$$1+\frac{\lambda-b}{\alpha} = -\int_{-h}^{0} a(s)e^{\lambda s}ds$$

Put $\lambda = x + yi$. For the real part we get

$$1+\frac{x-b}{\alpha} = -\int_{-h}^{0} a(s)e^{xs}\cos ys\,ds$$

We will show by contradiction that in this case x satisfies the inequality: $x \le -\omega_0 + b$.

So let us assume for instance that $-\omega_0 < x - b \le 0$. In that case we get

$$1-\varepsilon_0 = 1-\frac{\omega_0}{\alpha_0} < 1+\frac{x-b}{\alpha_0} \le$$

$$\le 1+\frac{x-b}{\alpha} = \left|\int_{-h}^{0} a(s)e^{xs}\cos ys\,ds\right| \le$$

$$\le e^{\omega_0 h}\int_{-h}^{0}|a(s)|ds = e^{\omega_0 h}\|a\|_{L^1(-h,0;R)} \le$$

$$\le e^{\omega_0 h}a_0 = 1-\varepsilon_0$$

which is a contradiction. Furthermore let us assume that $x - b > 0$. Then we get

$$1-\varepsilon_0 = 1-\frac{\omega_0}{\alpha_0} < 1 < 1+\frac{x-b}{\alpha_0} \le$$

$$\le \left| \int_{-h}^{0} a(s)e^{xs}\cos ys\,ds \right| \le$$

$$\le \int_{-h}^{0} |a(s)|\,ds = \|a\|_{L^1(-h,0;R)} \le a_0 < 1$$

which is a contradiction again. This shows that $x \le -\omega_0+b$. Moreover, since $b \le 0$ it follows that $x \le -\omega_0$.

Suppose that $\lambda \in \Gamma_0$. Then by (4.3) λ satisfies the equation

$$1+\int_{-h}^{0} a(s)e^{\lambda s}ds = 0$$

Assume that $\mathrm{Re}\lambda > -\omega_0$. In that case we get:

$$1 = \left| \int_{-h}^{0} a(s)e^{\lambda s}ds \right| \le$$

$$\le \int_{-h}^{0} |a(s)|e^{xs}ds \le e^{\omega_0 h}\int_{-h}^{0} |a(s)|\,ds =$$

$$= e^{\omega_0 h}\|a\|_{L^1(-h,0;R)} \le e^{\omega_0 h}a_0 = 1-\varepsilon_0$$

which is a contradiction. Thus for every $\lambda \in \sigma(\Lambda)$we have the following estimate: $\mathrm{Re}\lambda \le -\omega_0$, where ω_0 is an arbitrary number from the interval $(0, \alpha_0)$. By (4.6) for sufficiently small ω_0 the upper bound a_0 will attain any value close to and less than 1. This means that whenever

$$\|a\|_{L^1(-h,0;R)} \le a_0 < 1$$

there is an $\omega_0 > 0$ such that $\mathrm{Re}\lambda \le -\omega_0 < 0$ for every $\lambda \in \sigma(\Lambda)$. Therefore by relation (1.3) there exist $\omega > 0$ and $M > 0$ such that

$$\|T(t)\| \le Me^{-\omega t} \quad \text{for all } t \ge 0,$$

which completes the proof of Proposition 2.

EXAMPLE. The results given above can be applied to the following initial boundary value problem for a retarded partial integrodifferential equation:

$$\begin{aligned}
&u_t(t,x) = \Delta u(t,x) + bu(t,x) + \int_{-h}^{0} a(r)\Delta u(t+r,x)dr && \text{for a.e. } (t,x) \in [0,T]\times\Omega\\
&u(t,x) = g(t,x) && \text{for a.e. } (t,x) \in [-h,0]\times\Omega\\
&u(0,x) = f(x) && \text{for a.e. } x \in \Omega\\
&u(t,x) = 0 && \text{for a.e. } (t,x) \in [0,T]\times\partial\Omega
\end{aligned}$$

where Δ is a Laplacian, Ω is a open bounded subset of R^n with regular boundary $\partial\Omega$, $b \in R$, $a \in L^2(-h,0;R)$ and g and f are given functions. We can consider this problem as an initial value problem in the Hilbert space $X = L^2(\Omega)$ by setting $A = \Delta$ and $D(A) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. For the details see e.g. [9],[11].

COROLARY. If $b \le 0$ and a satisfies condition (4.4) then the solution $u = u(t,x)$ is asymptotically stable.

Proof. We note that A is a selfadjoint and negative operator with spectrum lying on the negative real axis and satisfying condition (4.1). It also generates a bounded analytic semigroup on X. Thus the asymptotic stability of u follows directly from Proposition 2.

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