

AN $M_2/G_2/1$ RETRIAL QUEUE WITH PRIORITY CUSTOMERS, 2ND OPTIONAL SERVICE AND LINEAR RETRIAL POLICY

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ABSTRACT:

We analyze an $M_2/G_2/1$ retrial queuing system with two types of customers and linear retrial policy. If any arriving customer finds the server idle, then it begins his service immediately. Blocked customers from the first flow are queued in order to be served; whereas blocked customers from the second flow leave the service area, but after some random amount of time they repeat an attempt to get service. After essential service completion, a customer either may abandon the system forever or may immediately ask for a second service. The essential and optional service times are arbitrarily and exponentially distributed respectively. We study the ergodicity of the embedded Markov chain, its stationary distribution function and the joint generating function of the number of customers in both groups in the steady-state regime.

Key words: Embedded Markov Chain, Ergodicity, Steady-state distribution, Regular and Optional Services.

MSC 60K25

RESUMEN

Consideremos el sistema de colas con reintentos $M_2/G_2/1$ con dos tipos de clientes y política de reintentos lineal. Si un cliente llega al sistema y encuentra el servidor libre, entonces empieza su servicio inmediatamente. Si no, los clientes del primer tipo van a una cola y apenas se libera el servidor, el primer cliente empieza su servicio. El segundo tipo de clientes al encontrar el servidor ocupado sale del área de servicio, pero luego de un tiempo aleatorio, regresa e intenta ser servido. Después de completar su servicio esencial, los clientes abandonan el sistema o acceden inmediatamente a un segundo servicio. Los tiempos de servicios esenciales son arbitrarios, los de los servicios optativos, exponencialmente distribuidos. En este trabajo estudiamos la ergodicidad de la cadena inducida de Markov, su función de distribución estacionaria y la función generadora conjunta del número de clientes en ambos grupos en el estado estacionario.

1. INTRODUCTION

Retrial queues are characterized by the feature that arriving customers who find the server busy, join the retrial group to try again for their request in random order and at random intervals. Retrial queues with two types of customers have been widely used to model and solve various practical problems occurring in telecommunication networks, telephone switching systems and digital cellular mobile networks. From the mathematical point of view, retrial queues with two types of customers are essentially more difficult than retrial queues with single type of customers. For a complete survey on single server retrial queues with two types of customers and their applications, see Choi *et al.* (1999).

For the retrial systems it is necessary to fix the mechanism of retrials. We will consider two different retrial policies for describing how the customers in orbit can access to the server. Most queuing systems with repeated attempts assume that each customer in orbit seeks service independently of each other after a random time exponentially distributed with rate γ . Thus, the probability of a repeated attempt during the interval $(t, t+\Delta t)$, given that j customers were in orbit at time t , is $j\gamma\Delta t + o(\Delta t)$. However, there is a second retrial policy where the retrial rate does not depend on the number of customers in orbit (if any), so the probability of a repeated attempt

during $(t, t+\Delta t)$, given that the orbit is not free at time t , is $\Delta t + o(\Delta t)$. This discipline was introduced by Fayolle (1986) and it is known as *constant retrial policy*. In what follows we will incorporate both retrial policies by assuming the *linear retrial policy* introduced in Artalejo-Gómez (1997); i.e., the probability of a repeated attempt during the interval $(t, t+\Delta t)$, given that j customers were in orbit at time t , is $((1-\delta_{0j})\alpha + j\gamma)\Delta t + o(\Delta t)$, where δ_{ij} denotes Kronecker's delta.

On the other hand, in day to day life, one encounters numerous examples of the queuing situations where all arriving customers require the main service and only some may require the subsidiary service provided by the server. For this reason we study a queuing system where the customers after the first (regular) service may choose for the second optional service or may leave the system forever. Thus, we specifically analyze a system where the customers' service may be scheduled in two phases; that is, all the customers are processed in the first phase and only the customers who qualify are routed to the second phase. As related works, the reader is referred to Doschi (1991), Krishna *et al.* (2001), Krishna *et al.* (2002), Madan *et al.* (2000) and Selvam *et al.* (1994).

In this paper, we analyze an $M_2/G_2/1$ retrial queue with priority customers, second optional service and linear retrial policy. The rest of the paper is organized as follows.

The remainder of the work is organized as follows. The next section gives a description of the queueing model. In section 3, we carry out the study of the embedded Markov chain at departure epochs, including its stationary distribution and ergodicity condition. Finally, in section 4, the joint distribution of the number of customers in the waiting line and the orbit is obtained in steady state.

2. DESCRIPTION OF THE QUEUE'S SYSTEM

Let's consider a queue's system with only one server and a priority buffer of infinite capacity. The clients may be of two classes, $i=1,2$, and they arrive to the system following independent Poisson's stream with intensities λ_i and λ_0 respectively. An h -demand will be a demand of class h with intensity's stream λ_h .

If the server is free in the moment of an external arrival, this demand begins to be served immediately and he abandons the system after completing its service. However, the behavior of a blocked demand (that is a demand that finds the server busy in the moment of their arrival) depends on their type. The 1-demands are placed in the high-priority queue, waiting to be served, while the blocked 2-demands, abandon the area of service, but after some random time they intend to get service. We suppose that the times among retries are independent and exponentially distributed with linear intensity $\gamma_j = (1-\delta_{0j})\alpha + j\gamma$ where j is the number of demands in the orbit. This way of access to the service from the orbit is known as *linear retry's policy*.

As soon as the first service of each h -demand is completed, with probability θ_h , it opts to receive its second service (that will begin immediately) and, with complementary probability $\bar{\theta}_h = 1 - \theta_h$, this demand will abandon the system. The times of the first service (essential services) of each h -demand, are general with an absolutely continuous distribution's function $B_h(x)$, with $B_h(0)=0$,

$B_{h,1} = \int_0^\infty x dB_h(x) < \infty$ and $B_h(s)$ the Laplace-Stieltjes' transformed. On the other hand, the

distribution's function of the second times of service (optional services) of the h -demands, is an exponential distribution, of mean $1/\mu_h$.

We will suppose that the flows of arrival of demands, the intervals between retries and the times of service are mutually independent.

In an arbitrary moment t the system can be described by the process

$$X(t) = (C(t), S(t), N_1(t), N_2(t), \xi(t))$$

where $C(t)$ denotes the state of the server 0 or h (0 if at moment t , the server is free, and h if it is occupied with an h -demand), $S(t)$ represents the type of service, 1 or 2 if in the moment t the server carries out an essential or optional service, $N_1(t)$ and $N_2(t)$ are the numbers of demands in the high-priority queue and in the orbit respectively. Finally $\xi(t)$ denotes the elapsed service's time of the demand that is being served. Naturally the process $S(t)$ is only defined when the server is busy, that is when $C(t) \in \{1, 2\}$. Due to the characteristics of the exponential distribution, the process $\xi(t)$ will be only defined if the server is busy with an essential service, when $S(t)=1$.

Because of the above description, it is clear that during the evolution of our system, the server will be busy or free alternatively. After each service, the server would be free if the priority queue is empty. In this case, a competition among three exponential laws with intensities λ_e , λ_\square and λ_j will determine that the following demand in being served, will be a j -demand or an element of the orbit. This is the main difference with the models without repeated demands.

It is easy to see that a 2-demand can only be admitted in the server if there are not demands in the high-priority queue. In this sense the 1-demands have priority on the 2-demands. That is why we will refer to the 1-demands as the priority-demands (of high priority) and to the 2-demands as the non high-priority demands (of low priority).

3. THE INDUCED MARKOV 'S CHAIN

Let η_l be the moment of the l^{th} exit (that is, the moment in which the l^{th} demands completes its service), $C_l = C_l(\eta_l - 0)$ the type of the l^{th} -demands served, $N_{1,l} = N_1(\eta_l - 0)$ and $N_{2,l} = N_2(\eta_l - 0)$ the number of demands in the high-priority queue and in the orbit exactly before the moment η_l . For $N_{1,l}$ and $N_{2,l}$ we have the following recursive equations.

$$N_{1,d} = \begin{cases} (N_{1,d-1} - 1)^+ + \nu_{1,d}^{(1)} + \nu_{1,d}^{(2)} & \text{with probability } \theta_{C_d} \\ (N_{1,d-1} - 1)^+ + \nu_{1,d}^{(1)} & \text{with probability } \check{\theta}_{C_d} \end{cases} \quad (1)$$

$$N_{1,d} = \begin{cases} N_{2,d-1} - B_d + \nu_{2,d}^{(1)} + \nu_{2,d}^{(2)} & \text{with probability } \theta_{C_d} \\ N_{2,d-1} - B_d + \nu_{1,d}^{(1)} & \text{with probability } \check{\theta}_{C_d} \end{cases} \quad (2)$$

where $\nu_{h,l}^{(1)}$ is the number of h -demands that arrive to the system during the time of essential service of the l^{th} -demand, $\nu_{h,l}^{(2)}$, is the number of h -demands arriving to the system during the time of optional service of the l^{th} -demands (in case it is received), and $B_l = 1$ if the l^{th} -demand served comes from the orbit, and 0 otherwise.

The random vector (C_l, B_l) depends on the history of the system before the moment η_{l-1} only through the vector $(N_{1,l-1}, N_{2,l-1})$, their conditional distribution comes given by the following formulas

$$\begin{aligned}
P\{(C_d, B_d) = (1,0) | (N_{1,d-1}, N_{2,d-1}) = (0, j)\} &= \frac{\lambda_1}{\lambda + \gamma_j} \\
P\{(C_d, B_d) = (1,0) | (N_{1,d-1}, N_{2,d-1}) = (i, j)\} &= 1 \quad \text{if } i \geq 1 \\
P\{(C_d, B_d) = (1,1) | (N_{1,d-1}, N_{2,d-1}) = (0, j)\} &= 0 \\
P\{(C_d, B_d) = (1,1) | (N_{1,d-1}, N_{2,d-1}) = (i, j)\} &= 0 \quad \text{if } i \geq 1 \\
P\{(C_d, B_d) = (2,0) | (N_{1,d-1}, N_{2,d-1}) = (0, j)\} &= \frac{\lambda_2}{\lambda + \gamma_j} \\
P\{(C_d, B_d) = (2,0) | (N_{1,d-1}, N_{2,d-1}) = (i, j)\} &= 0 \quad \text{if } i \geq 1 \\
P\{(C_d, B_d) = (2,1) | (N_{1,d-1}, N_{2,d-1}) = (0, j)\} &= \frac{\gamma_j}{\lambda + \gamma_j} \\
P\{(C_d, B_d) = (2,1) | (N_{1,d-1}, N_{2,d-1}) = (i, j)\} &= 0 \quad \text{if } i \geq 1
\end{aligned}$$

where $\lambda_1 + \lambda_2$

The random vector $(\nu_{1,l}^{(1)}, \nu_{2,l}^{(1)})$ depends on what has happened before the beginning of the l^{th} - demand's only through C_l . Its conditional distribution is given by

$$k_{h,i,j} = P\{(\nu_{1,l}^1, \nu_{2,l}^1) = (i, j) | C_l = h\} = \int_0^\infty \frac{\lambda_1 x^i}{i!} e^{-\lambda_1 x} \frac{\lambda_2 x^j}{j!} e^{-\lambda_2 x} dB_h(x)$$

and its generating function is

$$k_h(z_1, z_2) = E[z_1^{\nu_{1,l}^1}, z_2^{\nu_{2,l}^1} | C_l = h] = \beta_h \lambda_1 (1 - z_1) + \lambda_2 (1 - z_2)$$

for $h=1,2$.

The previous remarks, allow us to consider $X_l = (C_l, N_{1,l}, N_{2,l})$ as a Markov's chain with space of states $\{1,2\} \times N^2$. It is the induced Markov's chain of our system of lines. We can easily prove that this Markov's chain $\{X_l, l \in N\}$ is irreducible and aperiodic.

The transition's probabilities

$$p_{(k,m,n)(h,i,j)} = P\{X_l = (h,i,j) | X_{l-1} = (k,m,n)\}$$

are given by the formulas

$$\begin{aligned}
P_{(k,0,n)(1,i,j)} &= \frac{\lambda_1}{\lambda + \gamma_n} a_{1,i,j-n} & n = 0, 1, \dots, j \\
P_{(k,m,n)(1,i,j)} &= a_{1,i-m+1,j-n} & m = 1, \dots, j \\
P_{(k,0,n)(2,i,j)} &= \frac{\lambda_2}{\lambda + \gamma_n} a_{2,i,j-n} + \frac{\gamma_n}{\lambda + \gamma_n} a_{2,i,j-n+1} & n = 0, \dots, j \\
P_{(k,0,j+1)(2,i,j)} &= \frac{\gamma_{j+1}}{\lambda + \gamma_{j+1}} a_{2,i,0} \\
P_{(k,m,n)(2,i,j)} &= 0 & m \geq 1, n \geq 0
\end{aligned}$$

where $a_{h,i,j}$ is the combined distribution of the number of external demands of both types that arrive during the time of service of a h -demand and whose generating function is

$$a_h(z_1, z_2) = \ddot{\theta}_h k_h(z_1, z_2) + \square_h c_h(z_1, z_2), \quad h=1, 2.$$

As it is expected, at first we will study is the ergodicity of our induced Markov's chain. Due to the recursive structure of the equations (1) - (2) that describe the chain, we will use the Foster's approach.

In our case, we will consider $F(k, m, n) = am + n$ as test or Lyapunov's function on the space of states, where $a > 0$ is a constant to determine.

From equations (1) - (2) we obtain that

$$x_{k,m,n} = E[F(X_l) - F(X_{l-1}) \mid X_{l-1} = (k, m, n)]$$

satisfy

$$x_{k,m,n} = \begin{cases} \frac{a\lambda_1 + \lambda_2}{\lambda_2} \left(\frac{\lambda_2}{(\lambda + \gamma_n \rho_1)} \rho_1 + \frac{\lambda_2 + \gamma_n}{\lambda + \gamma_n} \rho_2 \right) - \frac{\gamma_n}{\lambda + \gamma_n} & \text{if } m = 0 \\ a(-1 + \rho_1) \frac{\lambda_2}{\lambda_1} \rho_1 & \text{if } m \geq 1 \end{cases}$$

where $\rho_h = \lambda_h(B_{h,1} + \theta_h/\mu_h)$ for $h=1, 2$.

Now we define the charge of the system (use's factor of the system) as $\rho = \rho_1 + \rho_2$ and we distinguish two cases.

If $\alpha \geq 0, \gamma > 0$.

Knowing that:

$$\begin{aligned} 0 &> \frac{a\lambda_1 + \lambda_2}{\lambda_2} \left(\frac{\lambda_2}{\lambda + \gamma_n} \rho_1 + \frac{\lambda_2 + \gamma_n}{\lambda + \gamma_n} \rho_2 \right) - \frac{\gamma_n}{\lambda + \gamma_n} \\ 0 &> a(-1 + \rho_1) + \frac{\lambda_2}{\lambda_1} \rho_1 \quad n \geq 0 \end{aligned}$$

are respectively equivalent, to:

$$a < \frac{\lambda_2}{\lambda_1} \left(\frac{1 - \rho_2}{\rho_2} \right) \text{ and } a > \frac{\rho_1}{1 - \rho_1} \frac{\lambda_2}{\lambda_1}$$

Such an a exists if and only if the interval

$$\left(\frac{\lambda_2}{\lambda_1} \frac{\rho_1}{1 - \rho_1}, \frac{\lambda_2}{\lambda_1} \frac{1 - \rho_2}{\rho_2} \right)$$

is non-empty, that is to say that if

$$\frac{\lambda_2}{\lambda_1} \frac{\rho_1}{1 - \rho_1} < \frac{\lambda_2}{\lambda_1} \frac{1 - \rho_2}{\rho_2} \Leftrightarrow \rho < 1$$

If $\alpha > 0, \gamma = 0$.

As the conditions

$$0 > \frac{a\lambda_1 + \lambda_2}{\lambda_2} \left(\frac{\lambda_2}{\lambda + \alpha} \rho_1 + \frac{\lambda_2 + \alpha}{\lambda + \alpha} \rho_2 \right) - \frac{\alpha}{\lambda + \alpha}$$

$$0 > a(-1 + \rho_1) + \frac{\lambda_2}{\lambda_1} \rho_1 \quad n \geq 1$$

are equivalent to:

$$a < \frac{\lambda_2}{\lambda_1} \left(\frac{\alpha(1 - \rho_2) - \lambda_2(\rho_1 + \rho_2)}{\alpha\rho_2 + \lambda_2(\rho_1 + \rho_2)} \right) \text{ and } a > \frac{\rho_1}{1 - \rho_1} \frac{\lambda_2}{\lambda_1}$$

the existence of the desired a is equivalent to the non-emptiness of the interval

$$\left(\frac{\rho_1}{1 - \rho_1} \frac{\lambda_2}{\lambda_1}, \frac{\lambda_2}{\lambda_1} \left(\frac{\alpha(1 - \rho_2) - \lambda_2(\rho_1 + \rho_2)}{\alpha\rho_2 + \lambda_2(\rho_1 + \rho_2)} \right) \right)$$

that is to say that

$$\frac{\rho_1}{1 - \rho_1} \frac{\lambda_2}{\lambda_1} < \frac{\lambda_2}{\lambda_1} \left(\frac{\alpha(1 - \rho_2) - \lambda_2(\rho_1 + \rho_2)}{\alpha\rho_2 + \lambda_2(\rho_1 + \rho_2)} \right) \Leftrightarrow \rho < \frac{\alpha}{\lambda_2 + \alpha}$$

Therefore, the condition

$$\rho < 1 - \frac{\lambda_2}{\lambda_2 + \alpha} \delta_{0,\gamma} \quad (3)$$

implies the ergodicity of the induced Markov's chain.

Later, we will see that the condition (3) is also necessary for its ergodicity.

Our next objective is to find the stationary distribution

$$\pi_{h,i,j} = \lim_{l \rightarrow \infty} P[X_l = (h, i, j)]$$

of the chain of induced Markov $\{X_l, l \in \mathbb{N}\}$.

Using the previous formulas for the transition probabilities of the induced Markov's chain, we obtain that the Kolmogorov's equations for $\pi_{h,i,j}$ are

$$\pi_{1,i,j} = \sum_{n=0}^j \pi_{0,n} \frac{\lambda_1}{\lambda + \gamma_n} a_{1,i,j-n} + \sum_{m=1}^{i+1} \sum_{n=0}^j \pi_{m,n} a_{1,i+1=m,j-n}$$

$$\pi_{2,i,j} = \sum_{n=0}^j \pi_{0,n} \frac{\lambda_2}{\lambda + \gamma_n} a_{2,i,j-n} + \sum_{n=1}^{j+1} \pi_{0,n} \frac{\gamma_n}{\lambda + \gamma_n} a_{2,i,j+1-n}$$

where $\pi_{i,j} = \pi_{1,i,j} + \pi_{2,i,j}$.

We introduce the generating functions

$$\begin{aligned}\varphi_h(z_1, z_2) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \pi_{h,i,j} z_1^i z_2^j \quad h = 1, 2. \\ \varphi(z_1, z_2) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \pi_{i,j} z_1^i z_2^j = \varphi_1(z_1, z_2) + \varphi_2(z_1, z_2)\end{aligned}$$

and the auxiliary generating function

$$\psi(z_2) = \sum_{j=0}^{\infty} \frac{\pi_{0,j}}{\lambda + \gamma_j} z_2^j$$

Using them, the Kolmogorov's equations are equivalent to:

$$z_1 \varphi_1(z_1, z_2) = a_1(z_1, z_2) [\lambda_1 z_1 \psi(z_2) + \varphi(z_1, z_2) - \varphi(0, z_2)] \quad (4)$$

$$\varphi_2(z_1, z_2) = a_2(z_1, z_2) \left[\lambda_2 \psi(z_2) + \frac{\alpha}{z_2} \left(\psi(z_2) - \frac{\pi(0,0)}{\lambda} \right) + \gamma \psi(z_2) \right] \quad (5)$$

Eliminating $\varphi_2(z_1, z_2)$ in the equations (4)-(5) and considering that

$$\varphi(0, z_2) = \gamma z_2 \psi'(z_2) + (\alpha + \gamma) \psi'(z_2) - \alpha \frac{\pi_{0,0}}{\lambda}$$

we obtain

$$\begin{aligned}z_2 \frac{z_1 - a_1(z_1, z_2)}{a_1(z_1, z_2)} \varphi_1(z_1, z_2) &= \alpha [z_2 - a_2(z_1, z_2)] \frac{\pi_{0,0}}{\lambda} + \gamma z_2 [a_2(z_1, z_2) - z_2] \psi'(z_2) - \\ &\quad - \lambda_1 z_2 (1 - z_1) + \lambda_2 z_2 [1 - a_2(z_1, z_2)] + \\ &\quad + \alpha [z_2 - a_2(z_1, z_2)] \psi(z_2)\end{aligned} \quad (6)$$

Let us consider the function

$$f(z_1, z_2) = z_1 - a_1(z_1, z_2).$$

For each fixed z_2 with $|z_2| < 1$, $f(z_1, z_2)$ is a function of z_1 . If $|z_1|=1$, then $\operatorname{Re}(\lambda - \lambda q z_1 - \lambda p z_2) > 0$. It is well-known that $|B(s)| < 1$ if $\operatorname{Re}(s) > 0$. Then we have

$$|f(z_1, z_2) - z_1| = |a_1(z_1, z_2)| \leq |k_1(z_1, z_2)| = |\beta_1(\lambda_1(1 - z_1) + \lambda_{-1}(1 - z_2))| < 1 = |z_1|.$$

By Rouché's theorem, the functions $f(z_1, z_2)$ and z_1 (that are analytic in the unit disk) have the same number of zeros in this disk. Now as z_1 has only a zero in the disk unit (that is the number 0), for each z_2 fixed, with $|z_2| < 1$, $f(z_1, z_2)$ has an unique zero in the unit disk. As a consequence, for each z_2 with $|z_2| < 1$ there is only one $z_1 = g(z_2)$ solution of the equation $f(z_1, z_2) = 0$ inside the circle unit, that is:

$$f(g(z_2), z_2) = g(z_2) - a_1(g(z_2), z_2) = 0.$$

It is easy to check that

$$\begin{aligned}(i) \quad &g(1) = 1 \\ (ii) \quad &g'(1) = \frac{\lambda_2}{\lambda_1} \frac{\rho_1}{1 - \rho_1}\end{aligned}$$

$$(iii) \quad g''(1) = \frac{\lambda_2^2}{(1-\rho_1)^3} \left[\frac{\theta_1}{\mu_1} \left(\beta_{1,1} + \frac{1}{\mu_1} \right) + \beta_{1,2} \right].$$

$$(iv) \quad \text{If } \rho < 1, \text{ then } a_2(g(z_2), z_2) = z_2 \Leftrightarrow z_2 = 1.$$

Making $z_1 = g(z_2)$ in (6) we obtain:

$$\begin{aligned} & \lambda_1 z_2 [1 - g(z_2)] + \lambda_2 z_2 [1 - a_2(g(z_2), z_2)] + \alpha [z_2 - a_2(g(z_2), z_2)] \psi(z_2) + \\ & \gamma z_2 [z_2 - a_2(g(z_2), z_2)] \psi'(z_2) = \alpha [z_2 - a_2(g(z_2), z_2)] \frac{\pi_{0,0}}{\lambda} \end{aligned} \quad (7)$$

On the other hand,

$$\lim_{z_2 \rightarrow 1} \frac{\lambda_1 z_2 [1 - g(z_2)] + \lambda_2 z_2 [1 - a_2(g(z_2), z_2)] + \alpha [z_2 - a_2(g(z_2), z_2)]}{z_2 - a_2(g(z_2), z_2)} = \frac{\alpha - (\lambda_2 + \alpha)\rho}{1 - \rho} < \infty \quad \text{if } \rho < 1$$

Therefore, the equation (7) can be written as

$$\frac{\lambda_1 z_2 [1 - g(z_2)] + \lambda_2 z_2 [1 - a_2(g(z_2), z_2)] + \alpha [z_2 - a_2(g(z_2), z_2)]}{z_2 - a_2(g(z_2), z_2)} \psi(z_2) + \gamma z_2 \psi'(z_2) = \alpha \frac{\pi_{0,0}}{\lambda} \quad (8)$$

where the coefficient of $\psi(z_2)$ is defined in $[0, 1)$ and in $z_2 = 1$ it is extended by continuity. Substituting $\psi'(z_2)$ by its value in the equations (5)-(6) we have:

$$\begin{aligned} \varphi_1(z_1, z_2) &= a_1(z_1, z_2) \frac{\psi(z_2) \{ [z_2 - a_2(z_1, z_2)] [\lambda_1 (1 - g(z_2)) + \lambda_2 (1 - a_2(g(z_2), z_2))] \}}{[z_1 - a_1(z_1, z_2)]^{-1} [z_2 - a_2(g(z_2), z_2)]^{-1}} - \\ &\quad - [z_2 - a_2(g(z_2), z_2)] \left[\lambda_1 (1 - z_1) + \lambda_2 (1 - a_2(z_1, z_2)) + \frac{\alpha}{z_2} [z_2 - a_2(z_1, z_2)] \right] \\ \varphi_2(z_1, z_2) &= a_2(z_1, z_2) + \frac{\lambda_1 [1 - g(z_2)] + \lambda_2 [1 - z_2]}{a_2(g(z_2), z_2) - z_2} \psi(z_2) \end{aligned}$$

Now from the normalization condition $\varphi(1, 1) = 1$ we can find the unknown constant :

$$\psi(1) = \frac{1 - \rho}{\lambda}$$

The differential equation (8) can be written as

$$\gamma z_2 \psi'(z_2) + \left(\alpha + \frac{\lambda_1 z_2 [1 - g(z_2)] + \lambda_2 z_2 [1 - a_2(g(z_2), z_2)]}{z_2 - a_2(g(z_2), z_2)} \right) \psi(z_2) = \alpha \frac{\pi_{0,0}}{\lambda} \quad (9)$$

To solve this equation we consider three cases.

$$(a) \quad \text{PURE LINEAR RETRY, } \alpha > 0, \gamma > 0.$$

The solution of the differential equation (9) is

$$\psi(z_2) = \exp\left(-\frac{1}{\gamma} \int_1^{z_2} \left[\frac{\alpha}{u} + \frac{\lambda_1[1-g(u)] + \lambda_2[1-a_2(g(u),u)]}{u-a_2(g(u),u)} \right] du\right) \times$$

$$\left\{ \frac{(1-\rho)}{\lambda} + \frac{\alpha}{\gamma} \frac{\pi_{0,0}}{\lambda} \int_1^{z_2} \frac{1}{u} \exp\left(\frac{1}{\gamma} \int_1^u \frac{\alpha}{v} + \frac{\lambda_1[1-g(v)] + \lambda_2[1-a_2(g(v),v)]}{v-a_2(g(v),v)} dv\right) du \right\}$$

from here

$$\psi(z_2) = z_2^{-\frac{\alpha}{\gamma}} \exp\left(-\frac{1}{\gamma} \int_1^{z_2} \left[\frac{\lambda_1[1-g(u)] + \lambda_2[1-a_2(g(u),u)]}{u-a_2(g(u),u)} \right] du\right) \times$$

$$\left\{ \frac{(1-\rho)}{\lambda} + \frac{\alpha}{\gamma} \frac{\pi_{0,0}}{\lambda} \int_1^{z_2} u^{\frac{\alpha}{\gamma}-1} \exp\left(\frac{1}{\gamma} \int_1^u \frac{\alpha}{v} + \frac{\lambda_1[1-g(v)] + \lambda_2[1-a_2(g(v),v)]}{v-a_2(g(v),v)} dv\right) du \right\} \quad (10)$$

As $\lim_{z_2 \rightarrow 0^+} \psi(z_2) = \frac{\pi_{0,0}}{\lambda} < \infty$ and since the function

$$\frac{\lambda_1[1-g(u)] + \lambda_2[1-a_2(g(u),u)]}{u-a_2(g(u),u)}$$

is continuous in $[0, 1]$, taking limits for $z_2 \rightarrow 0^+$ in (10) we obtain that

$$\frac{1-\rho}{\lambda} + \frac{\alpha}{\gamma} \frac{\pi_{0,0}}{\lambda} \int_1^0 u^{\frac{\alpha}{\gamma}-1} \exp\left(\frac{1}{\gamma} \int_1^u \frac{\lambda_1[1-g(v)] + \lambda_2[1-a_2(g(v),v)]}{v-a_2(g(v),v)} dv\right) du = 0$$

then

$$\pi_{0,0} = (1-\rho) \frac{\gamma}{\alpha} \left\{ \int_0^1 u^{\frac{\alpha}{\gamma}-1} \exp\left(\frac{1}{\gamma} \int_1^u \frac{\lambda_1[1-g(v)] + \lambda_2[1-a_2(g(v),v)]}{v-a_2(g(v),v)} dv\right) du \right\}^{-1}$$

(b) CLASSIC RETRY, $\alpha=0, \gamma>0$.

The differential equation (9) would be

$$\psi'(z_2) + \frac{1}{\gamma} \frac{\lambda_1[1-g(z_2)] + \lambda_2[1-a_2(g(z_2),z_2)]}{z_2-a_2(g(z_2),z_2)} \psi(z_2) = 0$$

It is a homogeneous differential equation whose solution is

$$\psi(z_2) = \frac{1-\rho}{\lambda} \exp\left(-\frac{1}{\gamma} \int_1^{z_2} \frac{\lambda_1[1-g(u)] + \lambda_2[1-a_2(g(u),u)]}{u-a_2(g(u),u)} du\right)$$

(c) CONSTANT RETRY, $\alpha>0, \gamma=0$.

In this case, from (9) we have

$$\psi(z_2) = \alpha \frac{\pi_{0,0}}{\lambda} \left(\alpha + \frac{\lambda_1 z_2 [1 - g(z_2)] + \lambda_2 z_2 [1 - a_2(g(z_2), z_2)]}{z_2 - a_2(g(z_2), z_2)} \right)^{-1}$$

Taking limits when $z_2 \rightarrow 1$

$$\frac{1-\rho}{\lambda} = \alpha \frac{\pi_{0,0}}{\lambda} \frac{1-\rho}{\alpha - (\lambda_2 + \alpha)\rho}$$

then

$$\pi_{0,0} = \frac{\alpha - (\lambda_2 + \alpha)\rho}{\alpha} \quad (11)$$

Now from (11), as $\pi_{0,0} > 0$, we have that

$$\rho < 1 - \frac{\lambda_2}{\lambda_2 + \alpha}$$

So it is, in the case of constant retry, a necessary condition for the ergodicity of the chain. }

Lemma 1: *If $\rho \geq 1$, the induced Markov's chain is not ergodic.*

Proof: If $\rho > 1$, then of (i)-(iii), the function $G(z_2) = a_2(g(z_2), z_2)$ verifies the properties $G(1) = 1$, $G'(1) > 1$ and $G''(1) > 0$. Therefore, in a neighborhood, to the left of 1, G is a convex, positive, strictly increasing function. Then, since $G(0) > 0$ exists $z_2^* \in (0, 1)$ such that $G(z_2^*) = z_2^*$, that is $a_2(g(z_2^*), z_2^*) = z_2^*$. Making $z_2 = z_2^*$ in (7) we arrive to $\psi(z_2^*) = 0$, that means, $\pi_{0,j} = 0$ and therefore the Kolmogorov's equations don't have a trivial solution.

If $\rho = 1$, then starting from (7) we have

$$\begin{aligned} \psi(1) &= \lim_{z_2 \rightarrow 1} \frac{\left[\frac{\alpha \pi_{0,0}}{\lambda} - \gamma \psi'(1) \right] [z_2 - a_2(g(z_2), z_2)]}{\lambda_1 [z_2 - a_2(g(z_2), z_2)]} \\ &= \left[\frac{\alpha \pi_{0,0}}{\lambda} - \gamma \psi'(1) \right] \frac{1-\rho}{\alpha - (\lambda_2 + \alpha)\rho} = 0 \end{aligned}$$

and again $\pi_{0,j} = 0$, completing the proof of Lemma 1.

Consequently, the condition (3) is also necessary for the ergodicity of the chain. \square

The results of this section can be summarized in the following theorem.

Theorem 1: *The induced Markov's chain, that is $\{X(t), t \geq 0\}$ at the moments of the demands' exit, is ergodic if and only if*

$$\rho < 1 - \frac{\lambda_2}{(\lambda_2 + \alpha)} \delta_{0,\gamma}$$

and the generating functions of the stationary distribution of the chain are

$$\varphi_1(z_1, z_2) = \frac{a_1(z_1, z_2) \left\{ \psi(z_2) \left[z_2 - a_2(z_1, z_2) \right] \left\{ \lambda_1 [1 - g(z_2)] + \lambda_2 [1 - a_2(g(z_2), z_2)] + \frac{\alpha [z_2 - a_2(g(z_2), z_2)]}{z_2} \right\} \right\}}{[z_1 - a_1(z_1, z_2)] [z_2 - a_2(g(z_2), z_2)]} -$$

$$- [z_2 - a_2(g(z_2), z_2)] \left\{ \lambda_1 [1 - z_1] + \lambda_2 (1 - a_2(z_1, z_2)) + \frac{\alpha [z_2 - a_2(z_1, z_2)]}{\lambda} \right\}$$

$$\varphi_2(z_1, z_2) = a_2(z_1, z_2) \frac{\lambda_1 [1 - g(z_2)] + \lambda_2 (1 - z_2)}{[a_2(g(z_2), z_2) - z_2]} \psi(z_2)$$

where $g(z_2)$ is the unique root of $z_1 - a_1(z_1, z_2) = 0$ and corresponding to the three cases previously displayed we have:

(a) In pure linear retry

$$\psi(z_2) = \frac{z_2^{-\frac{\alpha}{\gamma}}}{\exp} \left\{ - \frac{\int_1^{z_2} \frac{\lambda_1 [1 - g(u)] + \lambda_2 [1 - a_2(g(u), u)]}{u - a_2(g(u), u)} du}{\gamma} \right\} \times$$

$$\times \left\{ \frac{1 - \rho}{\lambda} + \frac{\alpha}{\gamma} \frac{\pi_{0,0}}{\lambda} \right\} \int_1^{z_2} u^{\frac{\alpha}{\gamma} - 1} \exp \left\{ \frac{\int_1^u \frac{\lambda_1 [1 - g(v)] + \lambda_2 [1 - a_2(g(v), v)]}{v - a_2(g(v), v)} dv}{\gamma} \right\} du$$

with

$$\pi_{0,0} = \frac{\gamma}{\alpha} \{1 - \rho\} \left\{ \int_1^{z_2} u^{\frac{\alpha}{\gamma} - 1} \exp \left\{ \frac{\int_1^u \frac{\lambda_1 [1 - g(v)] + \lambda_2 [1 - a_2(g(v), v)]}{v - a_2(g(v), v)} dv}{\gamma} \right\} du \right\}^{-1}$$

b) In classic retry

$$\psi(z_2) = \frac{1 - \rho}{\lambda} \exp \left\{ - \frac{\int_1^{z_2} \frac{\lambda_1 [1 - g(u)] + \lambda_2 [1 - a_2(g(u), u)]}{u - a_2(g(u), u)} du}{\gamma} \right\}$$

(c) In constant retry

$$\psi(z_2) = \frac{\alpha - (\lambda_2 + \alpha)\rho}{\lambda} \left\{ \alpha + \frac{\frac{\lambda_1 z_2 [1 - g(z_2)] + \lambda_2 z_2 [1 - a_2(g(z_2), z_2)]}{z_2 - a_2(g(z_2), z_2)}}{\gamma} \right\}^{-1}$$

4. ANALYSIS OF THE STATIONARY STATE PROBABILITIES

Let us consider the limit probabilities

$$\begin{aligned} p_0(0, j) &= \lim_{t \rightarrow \infty} P[C(t) = 0, N_1(t) = 0, N_2(t) = j] \\ p_h^{(1)}(x, i, j) &= \lim_{t \rightarrow \infty} P[C(t) = h, S(t) = 1, N_1(t) = 0, N_2(t) = j, x < \xi(t) \leq x + dx] \\ p_h^{(2)}(i, j) &= \lim_{t \rightarrow \infty} P[C(t) = h, S(t) = 2, N_1(t) = i, N_2(t) = j] \end{aligned}$$

Applying the supplementary variable's method, we can easily obtain the system of equilibrium's equations

$$(\lambda + \gamma_j) p_0(0, j) = \sum_{h=1}^2 \theta_h \int_0^\infty p_h^{(1)}(x, 0, j) b_h(x) dx + \sum_{h=1}^2 \mu_h p_h^{(2)}(0, j) \quad (12)$$

$$\begin{aligned} \frac{d}{dx} p_h^{(1)}(x, i, j) &= -(\lambda + b_h(x)) p_h^{(1)}(x, 0, j) + (1 - \delta_{0,i}) \lambda_2 p_h^{(1)}(x, i-1, j) + \\ &+ (1 - \delta_{0,j}) \lambda_2 p_h^{(1)}(x, i, j-1), \quad h = 1, 2, i \geq 0 \end{aligned} \quad (13)$$

$$\begin{aligned} p_1^{(1)}(0, i, j) &= \sum_{h=1}^2 \theta_h \int_0^\infty p_h^{(1)}(x, i+1, j) b_h(x) dx + \sum_{h=1}^2 \mu_h p_h^{(2)}(i+1, j) \\ &+ \delta_{0,i} \lambda_1 p_0(0, j), \quad i \geq 0 \end{aligned} \quad (14)$$

$$p_1^{(2)}(0, i, j) = \delta_{0,i} \lambda_2 p_0(0, j) + \delta_{0,i} \gamma_{j+1} p_0(0, j+1) \quad i \geq 0 \quad (15)$$

$$\begin{aligned} (\lambda + \mu_h) p_h^{(2)}(0, i, j) &= \theta_h \int_0^\infty p_h^{(1)}(x, i, j) b_h(x) dx + (1 - \delta_{0,i}) \lambda_1 p_h^{(2)}(i-1, j) + \\ &+ (1 - \delta_{0,j}) \lambda_2 p_h^{(2)}(i, j-1) \quad h = 1, 2, i \geq 0 \end{aligned} \quad (16)$$

for $j \geq 0$, where $b_h(x) = \frac{1}{1 - B_h(x)} \frac{d}{dx} B_h(x)$ is the intensity of finishing the service of a h - demand in the moment x .

To solve the previous system we introduce the following generating functions

$$\begin{aligned} P_0(z_2) &= \sum_{j=0}^{\infty} p_0(0, j) z_2^j \\ P_h^{(1)}(x, z_1, z_2) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p_h^{(1)}(x, i, j) z_1^i z_2^j \quad h = 1, 2 \\ P_h^{(2)}(z_1, z_2) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p_h^{(2)}(i, j) z_1^i z_2^j \quad h = 1, 2. \end{aligned}$$

We will denote as

$$p_h^{(1)}(i, j) = \int_0^\infty p_h^{(1)}(x, i, j) dx$$

the probability that there are i demands in the high-priority queue, j demands in the orbit and the server is making an essential service to a h -demand, and we will denote the corresponding generating function as:

$$P_h^{(1)}(z_1, z_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p_h^{(1)}(i, j) z_1^i z_2^j$$

It is clear that

$$P_h^{(1)}(z_1, z_2) = \int_0^\infty P_h^{(1)}(x, z_1, z_2) dx$$

Using the previous generating functions, the system (12) - (16) is transformed in

$$\gamma z_2 P_0'(z_2) + (\lambda + \alpha) P_0(z_2) = \alpha p_0(0,0) + \sum_{h=1}^2 \theta_h \int_0^\infty P_h^{(1)}(x, 0, z_2) b_h(x) dx + \sum_{h=1}^2 \mu_h P_h^{(2)}(0, z_2) \quad (17)$$

$$\frac{\partial}{\partial x} P_h^{(1)}(x, z_1, z_2) = (\lambda_1(1 - z_1) + \lambda_2(1 - z_2) + b_h(x)) P_h^{(1)}(x, z_1, z_2) \quad h = 1, 2 \quad (18)$$

$$z_1 P_1^{(1)}(0, z_1, z_2) = \sum_{h=1}^2 \theta_h \int_0^\infty [P_h^{(1)}(x, z_1, z_2) - P_h^{(1)}(x, 0, z_2)] b_h(x) dx + \sum_{h=1}^2 \mu_h [P_h^{(2)}(z_1, z_2) - P_h^{(2)}(0, z_2)] + \lambda_1 z_1 P_0(z_2) \quad (19)$$

$$z_2 P_2^{(1)}(0, z_1, z_2) = \gamma z_2 P_0'(z_2) + (\lambda_2 z_2 + \alpha) P_0(z_2) - \alpha p_0(0,0) + \delta_{0i} \lambda_1 p_0(0, j), \quad i \geq 0 \quad (20)$$

$$\theta_h \int_0^\infty P_h^{(1)}(x, z_1, z_2) b_h(x) dx = (\lambda_1(1 - z_1) + \lambda_2(1 - z_2) + \mu_h) P_h^{(2)}(z_1, z_2) + (1 - \delta_{0i}) \lambda_1 p_h^{(2)}(i - 1, j) \quad h = 1, 2. \quad (21)$$

The solution of the differential equation (18) is

$$P_h^{(1)}(x, z_1, z_2) = P_h^{(1)}(0, z_1, z_2) [1 - B_h(x)] e^{-[\lambda_1(1 - z_1) + \lambda_2(1 - z_2)]x} \quad h = 1, 2 \quad (22)$$

Substituting (22) in (17), (19) and (21) we have that

$$\gamma z_2 P_0'(z_2) + (\lambda + \alpha) P_0(z_2) = \alpha p_0(0,0) + \sum_{h=1}^2 \check{\theta}_h k_h(0, z_2) P_h^{(1)}(0,0, z_2) + \sum_{h=1}^2 \mu_h P_h^{(2)}(0, z_2) \quad (23)$$

$$\begin{aligned} & \lambda_1 z_1 P_0(z_2) + \check{\theta}_2 k_2(z_1, z_2) P_1^{(2)}(0, z_1, z_2) + \\ [z_1 - \check{\theta}_1 k_1(z_1, z_2)] P_1^{(1)}(0, z_1, z_2) &= + \sum_{h=1}^2 \mu_h [P_h^{(2)}(z_1, z_2) - P_h^{(2)}(0, z_2)] - \sum_{h=1}^2 \check{\theta}_h k_h(0, z_2) P_h^{(1)}(0,0, z_2) \end{aligned} \quad (24)$$

$$\check{\theta}_h k_h(z_1, z_2) P_h^{(1)}(0, z_1, z_2) = (\lambda_1(1 - z_1) + \lambda_2(1 - z_2) + \mu_h) P_h^{(2)}(z_1, z_2) \quad (25)$$

Adding (23) - (24) we have

$$\begin{aligned} & \gamma z_2 P_0'(z_2) + (\lambda_1(1 - z_1) + \lambda_2 + \alpha) P_0(z_2) + [z_1 - \check{\theta}_1 k_1(z_1, z_2)] P_1^{(1)}(0, z_1, z_2) = \\ & = \alpha p_0(0,0) + \check{\theta}_2 k_2(z_1, z_2) P_1^{(2)}(0, z_1, z_2) + \sum_{h=1}^2 \mu_h P_h^{(2)}(0, z_2) \end{aligned} \quad (26)$$

Eliminating $P_h^{(2)}(z_1, z_2)$ of (25) - (26) we obtain

$$\begin{aligned} & \gamma z_2 P_0'(z_2) + (\lambda_1(1 - z_1) + \lambda_2 + \alpha) P_0(z_2) + [z_1 - a_1(z_1, z_2)] P_1^{(1)}(0, z_1, z_2) = \\ & = \alpha p_0(0,0) + a_2(z_1, z_2) P_2^{(1)}(0, z_1, z_2) \end{aligned}$$

Substituting (20) in the previous equation, we have

$$\begin{aligned} & z_2 [z_1 - a_1(z_1, z_2)] P_1^{(1)}(0, z_1, z_2) + \gamma z_2 [z_2 - a_2(z_1, z_2)] P_0'(z_2) + \\ & + \{ \lambda_1 z_2 (1 - z_1) + \lambda_2 z_2 [1 - a_2(z_1, z_2)] + \alpha [z_2 - a_2(z_1, z_2)] \} P_0(z_2) = \\ & = \alpha [z_2 - a_2(z_1, z_2)] p_0(0,0) \end{aligned} \quad (27)$$

Making $z_1 = g(z_2)$ in (27) we get

$$\begin{aligned} & \{ \lambda_1 z_2 (1 - g(z_2)) + \lambda_2 z_2 [1 - a_2(g(z_2), z_2)] + \alpha [z_2 - a_2(g(z_2), z_2)] \} P_0(z_2) + \\ & + \gamma z_2 [z_2 - a_2(z_1, z_2)] P_0'(z_2) = \alpha [z_2 - a_2(z_1, z_2)] p_0(0,0) \end{aligned}$$

As it have been done in the previous section, we can write

$$\gamma z_2 P_0'(z_2) + \left\{ \alpha + \frac{\lambda_1 z_2 [1 - g(z_2)] + \lambda_2 z_2 [1 - a_2(g(z_2), z_2)]}{z_2 - a_2(g(z_2), z_2)} \right\} P_0(z_2) = \alpha p_0(0,0) \quad (28)$$

where the coefficient of $P_0(z_2)$ is defined in $[0, 1)$ and in $z_2 = 1$ the function is extended by continuity.

From (27) - (28) we can find

$$\begin{aligned} \lambda z_2 P_1^{(1)}(0, z_1, z_2) = & \frac{P_0(z_2)}{[z_1 - a_1(z_1, z_2)][z_2 - a_2(g(z_2), z_2)]} \times \\ & \left\{ [z_2 - a_2(z_1, z_2)][\lambda_1[1 - g(z_2)] + \lambda_2[1 - a_2(g(z_2), z_2)]] + \frac{\alpha}{z_2[z_2 - a_2(g(z_2), z_2)]} \right\} - \\ & - \left\{ [z_2 - a_2(g(z_2), z_2)][\lambda_1[1 - z_1] + \lambda_2[1 - a_2(z_1, z_2)]] + \frac{\alpha}{z_2[z_2 - a_2(z_1, z_2)]} \right\} \end{aligned} \quad (29)$$

Using (20) and (28) we obtain

$$P_2^{(1)}(0, z_1, z_2) = \frac{\lambda_1[1 - g(z_2)] + \lambda_2(1 - z_2)}{a_2(g(z_2), z_2) - z_2} P_0(z_2) \quad (30)$$

(22) and (29) imply

$$\begin{aligned} P_1^1(x, z_1, z_2) = & \frac{[1 - B_1(x)]e^{-[\lambda_1[1 - z_2] + \lambda_2(1 - z_2)]x} P_0(z_2)}{[z_1 - a_1(z_1, z_2)][z_2 - a_2(g(z_2), z_2)]} \times \\ & \left\{ [z_2 - a_2(z_1, z_2)] \left\{ \lambda_1[1 - g(z_2)] + \lambda_2[1 - a_2(g(z_2), z_2)] + \frac{\alpha}{z_2} [z_2 - a_2(g(z_2), z_2)] \right\} - \right. \\ & \left. [z_2 - a_2(g(z_2), z_2)] \left\{ \lambda_1[1 - z_1] + \lambda_2[1 - a_2(z_1, z_2)] + \frac{\alpha}{z_2} [z_2 - a_2(z_2, z_2)] \right\} \right\} \end{aligned}$$

(22) and (30) lead us to

$$P_2^{(1)}(x, z_1, z_2) = \frac{\lambda_1[1 - g(z_2)] + \lambda_2(1 - z_2)}{a_2(g(z_2), z_2) - z_2} [1 - B_2(x)] e^{\{=[\lambda_1[1 - g(z_2)] + \lambda_2(1 - z_2)]x\}} P_0(z_2)$$

Of (25) and (29) we can find

$$\begin{aligned} P_1^{(2)}(z_1, z_2) = & \frac{\theta_1 k_1(z_1, z_2) P_0(z_2)}{[z_1 - a_1(z_1, z_2)][z_2 - a_2(g(z_2), z_2)][\lambda_1(1 - z_1) + \lambda_2(1 - z_2) + \mu_1]} \times \\ & \left\{ [z_2 - a_2(z_1, z_2)] \left\{ \lambda_1[1 - g(z_2)] + \lambda_2[1 - a_2(g(z_2), z_2)] + \frac{\alpha}{z_2} [z_2 - a_2(g(z_2), z_2)] \right\} - \right. \\ & \left. [z_2 - a_2(g(z_2), z_2)] \left\{ \lambda_1[1 - z_1] + \lambda_2[1 - a_2(z_1, z_2)] + \frac{\alpha}{z_2} [z_2 - a_2(z_2, z_2)] \right\} \right\} \end{aligned}$$

And from (25) and (30):

$$P_2^{(2)}(z_1, z_2) = \frac{\theta_2 k_2(z_1, z_2) [\lambda_1(1 - g(z_2)) + \lambda_2(1 - z_2)]}{[a_2(g(z_2), z_2) - z_2][\lambda_1(1 - z_1) + \lambda_2(1 - z_2) + \mu_2]} P_0(z_2)$$

If we neglect the time of lapsed service $\square(t)$, when $S(t)=1$ then:

$$P_1^{(1)}(z_1, z_2) = \frac{(1 - k_1(z_1, z_2))P_0(z_2)}{[z_1 - a_1(z_1, z_2)][z_2 - a_2(g(z_2), z_2)][\lambda_1(1 - z_1) + \lambda_2(1 - z_2)]} \times$$

$$\left\{ [z_2 - a_2(z_1, z_2)] \left\{ \lambda_1[1 - g(z_2)] + \lambda_2[1 - a_2(g(z_2), z_2)] + \frac{\alpha}{z_2} [z_2 - a_2(g(z_2), z_2)] \right\} - \right.$$

$$\left. [z_2 - a_2(g(z_2), z_2)] \left\{ \lambda_1[1 - z_1] + \lambda_2[1 - a_2(z_1, z_2)] + \frac{\alpha}{z_2} [z_2 - a_2(z_2, z_2)] \right\} \right\}$$

$$P_2^{(1)}(z_1, z_2) = \frac{(1 - k_2(z_1, z_2))[\lambda_1(1 - g(z_2)) + \lambda_2(1 - z_2)]}{[a_2(g(z_2), z_2) - z_2][\lambda_1(1 - z_1) + \lambda_2(1 - z_2)]} P_0(z_2)$$

Finally by the normalizing condition

$$P_0(1) + \square_{\square\square} P_h^{(1)}(1, 1) + \square_{\square\square} P_h^{(2)}(1, 1) = 1$$

$$P_0(1) = 1 - \square$$

Now, the differential equation (28), have to be solved and we will consider three cases.

(a) *PURE LINEAR RETRY*, $\alpha > 0$, $\gamma > 0$.

The solution of the differential equation (28) is

$$P_0(z_2) = \exp \left\{ -\frac{1}{\gamma} \int_1^{z_2} \left[\frac{\alpha}{u} + \frac{\lambda_1[1 - g(u)] + \lambda_2[1 - a_2(g(u), u)]}{u - a_2(g(u), u)} \right] du \right\} \times$$

$$\times \left\{ 1 - \rho + \alpha \frac{p_0(0,0)}{\gamma} \int_1^{z_2} \frac{1}{u} \exp \left\{ \frac{1}{\gamma} \int_1^u \left[\frac{\alpha}{v} + \frac{\lambda_1[1 - g(v)] + \lambda_2[1 - a_2(g(v), v)]}{v - a_2(g(v), v)} \right] dv \right\} du \right\}$$

from where

$$P_0(z_2) = z_2^{-\frac{\alpha}{\gamma}} \exp \left\{ -\frac{1}{\gamma} \int_1^{z_2} \left[\frac{\lambda_1[1 - g(u)] + \lambda_2[1 - a_2(g(u), u)]}{u - a_2(g(u), u)} \right] du \right\} \times$$

$$\times \left\{ 1 - \rho + \alpha \frac{p_0(0,0)}{\gamma} \int_1^{z_2} u^{\frac{\alpha}{\gamma}-1} \exp \left\{ \frac{1}{\gamma} \int_1^u \left[\frac{\lambda_1[1 - g(v)] + \lambda_2[1 - a_2(g(v), v)]}{v - a_2(g(v), v)} \right] dv \right\} du \right\} \quad (31)$$

As $\lim_{z_2 \rightarrow 0^+} P_0(z_2) = p_0(0,0) < \infty$ and

$$\frac{\lambda_1[1 - g(u)] + \lambda_2[1 - a_2(g(u), u)]}{1 - a_2(g(u), u)}$$

is a continuous function in $[0, 1]$, taking limits when $z_2 \rightarrow 0^+$ in (31) we obtain

$$\left\{ 1 - \rho + \alpha \frac{p_0(0,0)}{\gamma} \int_1^0 u^{\frac{\alpha}{\gamma}-1} \exp \left\{ \frac{1}{\gamma} \int_1^u \left[\frac{\lambda_1[1 - g(v)] + \lambda_2[1 - a_2(g(v), v)]}{v - a_2(g(v), v)} \right] dv \right\} du \right\} = 0$$

and

$$p_0(0,0) = \frac{(1-\rho)\gamma}{\alpha \left\{ \int_0^1 u^{\frac{\alpha}{\gamma}-1} \exp \left\{ \frac{1}{\gamma} \int_1^u \left[\frac{\lambda_1[1-g(v)] + \lambda_2[1-a_2(g(v),v)]}{v-a_2(g(v),v)} \right] dv \right\} du \right\}}$$

(b) *CLASSIC RETRY*, $\alpha=0, \gamma>0$.

The differential equation (28)

$$P_0'(z_2) = -\frac{1}{\gamma} \frac{\lambda_1[1-g(z_2)] + \lambda_2[1-a_2(g(z_2), z_2)]}{z_2 - a_2(g(z_2), z_2)} P_0(z_2)$$

is a homogeneous differential equation whose solution is

$$P_0(z_2) = (1-\rho) \exp \left\{ -\frac{1}{\gamma} \int_1^{z_2} \frac{\lambda_1[1-g(u)] + \lambda_2[1-a_2(g(u), u)]}{u - a_2(g(u), u)} du \right\}$$

(c) *CONSTANT RETRY*, $\alpha>0, \gamma=0$.

In this case, from (28) we have

$$P_0(z_2) = -\frac{\alpha p_0(0,0)}{\alpha + \frac{\lambda_1 z_2[1-g(z_2)] + \lambda_2 z_2[1-a_2(g(z_2), z_2)]}{z_2 - a_2(g(z_2), z_2)}}$$

Taking limits in the previous equation when $z_2 \rightarrow 1$ we obtain

$$1-\rho = \alpha p_0(0,0) \frac{1-\rho}{\alpha - (\lambda_2 + \alpha)\rho}$$

with

$$p_0(0,0) = \frac{\alpha - (\lambda_2 + \alpha)\rho}{\alpha} 1 - \frac{(\lambda_2 + \alpha)}{\alpha} \rho \quad (32)$$

In the case of constant retry, we can observe that from (32) and $p(0,0)>0$, the inequality $\rho < 1 - \lambda_2/(\lambda_1 + \alpha)$ must hold if the system is stable.

The results of this section can be summarized in the following theorem.

Theorem 2: *The stationary distribution of the process $\{X(t), t \geq 0\}$ is given by the following generating functions*

$$P_1^1(x, z_1, z_2) = \frac{[1 - B_1(x)]e^{-[\lambda_1[1-z_1]+\lambda_2(1-z_2)]x}P_0(z_2)}{[z_1 - a_1(z_1, z_2)][z_2 - a_2(g(z_2), z_2)]} \times \left\{ \begin{aligned} & [z_2 - a_2(z_1, z_2)] \left\{ \lambda_1[1 - g(z_2)] + \lambda_2[1 - a_2(g(z_2), z_2)] + \frac{\alpha}{z_2} [z_2 - a_2(g(z_2), z_2)] \right\} - \\ & [z_2 - a_2(g(z_2), z_2)] \left\{ \lambda_1[1 - z_1] + \lambda_2[1 - a_2(z_1, z_2)] + \frac{\alpha}{z_2} [z_2 - a_2(z_2, z_2)] \right\} \end{aligned} \right\}$$

$$P_2^{(1)}(x, z_1, z_2) = \frac{\lambda_1[1 - g(z_2)] + \lambda_2(1 - z_2)}{a_2(g(z_2), z_2) - z_2} [1 - B_2(x)]e^{\{-[\lambda_1[1 - g(z_2)] + \lambda_2(1 - z_2)]x\}} P_0(z_2)$$

$$P_1^{(2)}(z_1, z_2) = \frac{\theta_1 k_1(z_1, z_2) P_0(z_2)}{[z_1 - a_1(z_1, z_2)][z_2 - a_2(g(z_2), z_2)][\lambda_1(1 - z_1) + \lambda_2(1 - z_2) + \mu_1]} \times \left\{ \begin{aligned} & [z_2 - a_2(z_1, z_2)] \left\{ \lambda_1[1 - g(z_2)] + \lambda_2[1 - a_2(g(z_2), z_2)] + \frac{\alpha}{z_2} [z_2 - a_2(g(z_2), z_2)] \right\} - \\ & [z_2 - a_2(g(z_2), z_2)] \left\{ \lambda_1[1 - z_1] + \lambda_2[1 - a_2(z_1, z_2)] + \frac{\alpha}{z_2} [z_2 - a_2(z_2, z_2)] \right\} \end{aligned} \right\}$$

$$P_2^{(2)}(z_1, z_2) = \frac{\theta_2 k_2(z_1, z_2) [\lambda_1(1 - g(z_2)) + \lambda_2(1 - z_2)]}{[a_2(g(z_2), z_2) - z_2][\lambda_1(1 - z_1) + \lambda_2(1 - z_2) + \mu_2]} P_0(z_2)$$

where $g(z_2)$ is the only root of z_1 of the equation $z_1 - a_1(z_1, z_2) = 0$. Besides in the three cases previously displayed we have:

a) If $\alpha > 0, \gamma > 0$ (pure linear retry)

$$P_0(z_2) = z_2^{-\frac{\alpha}{\gamma}} \exp \left\{ -\frac{1}{\gamma} \int_1^{z_2} \left[\frac{\lambda_1[1 - g(u)] + \lambda_2[1 - a_2(g(u), u)]}{u - a_2(g(u), u)} \right] du \right\} \times \left\{ 1 - \rho + \alpha \frac{p_0(0,0)}{\gamma} \int_1^{z_2} u^{\frac{\alpha}{\gamma}-1} \exp \left\{ \frac{1}{\gamma} \int_1^u \left[\frac{\lambda_1[1 - g(v)] + \lambda_2[1 - a_2(g(v), v)]}{v - a_2(g(v), v)} \right] dv \right\} du \right\}$$

with

$$p_0(0,0) = \frac{(1 - \rho)\gamma}{\alpha \left\{ \int_0^1 u^{\frac{\alpha}{\gamma}-1} \exp \left\{ \frac{1}{\gamma} \int_1^u \left[\frac{\lambda_1[1 - g(v)] + \lambda_2[1 - a_2(g(v), v)]}{v - a_2(g(v), v)} \right] dv \right\} du \right\}}$$

b) If $\alpha = 0, \gamma > 0$ (classic retry)

$$P_0(z_2) = (1 - \rho) \exp \left\{ -\frac{1}{\gamma} \int_1^{z_2} \frac{\lambda_1[1 - g(u)] + \lambda_2[1 - a_2(g(u), u)]}{u - a_2(g(u), u)} du \right\}$$

c) If $\alpha > 0, \gamma = 0$ (constant retry)

$$P_0(z_2) = - \frac{\alpha - (\lambda_2 + \alpha)\rho}{\alpha + \frac{\lambda_1 z_2 [1 - g(z_2)] + \lambda_2 z_2 [1 - a_2(g(z_2), z_2)]}{z_2 - a_2(g(z_2), z_2)}}$$

If, when $S(t)=1$, we don't consider the elapsed time of $\xi(t)$, then the corresponding generating functions are:

$$P_1^1(z_1, z_2) = \frac{[1 - k_1(z_1, z_2)]P_0(z_2)}{[\lambda_1(1 - z_1) + \lambda_2(1 - z_2)][z_1 - a_1(z_1, z_2)][z_2 - a_2(g(z_2), z_2)]} \times$$

$$\left\{ [z_2 - a_2(z_1, z_2)] \left\{ \lambda_1[1 - g(z_2)] + \lambda_2[1 - a_2(g(z_2), z_2)] + \frac{\alpha}{z_2} [z_2 - a_2(g(z_2), z_2)] \right\} - \right.$$

$$\left. [z_2 - a_2(g(z_2), z_2)] \left\{ \lambda_1[1 - z_1] + \lambda_2[1 - a_2(z_1, z_2)] + \frac{\alpha}{z_2} [z_2 - a_2(z_2, z_2)] \right\} \right\}$$

$$P_1^1(z_1, z_2) = \frac{\left\{ \lambda_1[1 - g(z_2)] + \lambda_2[1 - z_2] + \frac{\alpha}{z_2} [z_2 - a_2(g(z_2), z_2)] \right\} [1 - k_2(z_1, z_2)] P_0(z_2)}{[\lambda_1(1 - z_1) + \lambda_2(1 - z_2)][a_2(g(z_2), z_2) - z_2]}$$

Corollary 1

1. The generating function of the stationary distribution of N_1 , (the number of demands in the high-priority line) is

$$Q(z) = P_0(1) + \sum_{h=1}^2 (P_h^{(1)}(z, 1) + P_h^{(2)}(z, 1))$$

2. The generating function of the stationary distribution of N_2 (the number of demands in the orbit) is

$$R(z) = P_0(z) + \sum_{h=1}^2 (P_h^{(1)}(1, z) + P_h^{(2)}(1, z))$$

3. The generating function of the stationary distribution of the combined variable (N_1+1, N_2) is

$$S(z_1, z_2) = P_0(z_2) + z_1 \sum_{h=1}^2 (P_h^{(1)}(z_1, z_2) + P_h^{(2)}(z_1, z_2))$$

4. The generating function of the stationary distribution of N , (the number of demands in the system) is

$$T(z) = P_0(z) + \sum_{h=1}^2 (P_h^{(1)}(z, z) + P_h^{(2)}(z, z))$$

Corollary 2

1. The probability of having the server free is

$$P_0(1)=1-\rho$$

2. The probability of finding the server busy is

$$1-P_0(1)=\rho$$

3. A high-priority demand is being (implied traffic of priority) with probability

- $$P_1^{(1)}(1,1) + P_1^{(2)}(1,1) = \rho_1$$
4. A low priority demand is being served (implied traffic of low priority) with probability
- $$P_2^{(1)}(1,1) + P_2^{(2)}(1,1) = \rho_2$$
5. The server would be busy with an essential service with probability
- $$P_1^{(1)}(1,1) + P_2^{(1)}(1,1) = \lambda_1 B_{1,1} + \lambda_2 B_{2,1}$$
6. The server would be busy with an optional service with probability
- $$P_1^{(2)}(1,1) + P_2^{(2)}(1,1) = \lambda_1 \theta_1 / \bar{\mu}_1 + \lambda_2 \theta_2 / \mu_2$$

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