MAXIMUM BALANCE CRITERION FOR CHOOSING THE PARAMETER λ IN THE MINIMAL PSEUDOINVERSE METHOD*

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ABSTRACT
We consider the problem of the numerical calculation of the pseudoinverse of an ill-conditioned matrix with ill-determined numerical rank. Basic concepts and statements of the Minimal Pseudoinverse Matrix method (MPM method) are resumed. The main purpose of this paper is to propose a new criterion, called *Maximum Balance*, which establishes a technique for choosing the parameter value λ in the MPM method when no accurate estimate of the level of perturbation in the given matrix is avaible. Also the numerical experimentation carried out is presented and discussed.

Key words: ill-posed problems, regularization methods, TSVD.

RESUMEN
En el artículo se considera el problema del cálculo numérico de la Seudoinversa o Inversa Generalizada de Moore-Penrose de una matriz mal condicionada con rango numérico mal determinado. Se resumen los aspectos básicos del método de la Matriz Seudoinversa Minimal (MSM) y se propone un nuevo criterio, llamado *Máximo Balance*, que establece una técnica para elegir el parámetro λ en el método MSM cuando no se dispone de información exacta sobre el nivel de perturbación de los elementos de la matriz. Se presenta y discute la experimentación numérica realizada.

MSC: 65.F20.

1. INTRODUCTION

This paper is concerned with the numerical calculation of the pseudoinverse of a matrix $A \in \Re^{mxn}$, which is known by means of the perturbed matrix $A_h \in \Re^{mxn}$ $\|A-A_h\| \leq h$, and $h > 0$ (here, $\|.\|$ denotes the Frobenius matrix norm). We also assume that A_h is an ill-conditioned matrix with ill-determined rank, i.e. the condition number is large and all its singular values decay to zero in such a way that there is no particular gap in the singular value spectrum.

Such problem, which arises in a variety of applications (astronomy, computational physics, engineering problems), is of a special interest in Numerical Analysis because it is considered,in the general case, an ill-posed problem in the Hadamard sense [López, J., (1986)]. Likewise, the problem of the calculation of the pseudoinverse of a matriz is very important when one wants to solve many linear systems with the same matrix and different right hand-sides. Such situation is often encountered in problems of processing experimental data obtained on an instrument.

The application of the truncation techniques based on the Singular Value Decomposition [Golub, G., (1989)], or RRQR decomposition [López, J. - Guerra, V. (1995)], is not recommended in our ill-determined rank case under discussion. So, more sophisticated methods are needed, one of them being the Minimal Pseudoinverse

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Matrix Method (MPM Method) ([Leonov, A., (1985)], (1987), (1991), (1992)], which is based on a variational regularization of the pseudoinverse A^+ that has important optimal properties [Leonov, A., (1992), [López, J. - Guerra, V., (1996)].

In the development of the MPM Method it is assumed that the level of perturbation h is a fairly accurate estimate. The exact knowledge of h is very important for choosing the "regularization" parameter $\lambda = \lambda(h)$. However, if only a rough estimate of h is known or if it is completely unknown, then the application of that method can be fatal.

In this work, we develop a new a posteriori strategy for choosing the parameter λ in the MPM method when no good estimation of the level of perturbation h is available. That technique is called *Maximum Balance Criterion*, and it consists in the calculation of a parameter λ in order to guarantee a good balance between the norm of the minimal pseudoinverse $\tilde{A}_h^+$ and the distance of the minimal matrix $\tilde{A}_h^+$ to A_h. That is reason for which we propose the construct ion and analysis of a parametric curve defined for all valid parameter values, and we present a new practical algorithm, which uses a conic for fitting to the discrete points.

Our criterion is inspired by the L-curve method for choosing the regularization parameter in the solution of ill-conditioned algebraic systems of linear equations [Hansen, P.C., (1992)], [Hansen, P.C., (1993)]. This method was proposed by Hansen in 1992 and it is based on heuristic approaches.

The organization of this paper is as follows: in section 2 we review the essential aspects of the MPM Method and the explicit form of the minimal pseudoinverse matrix. In the following section, we investigate various important properties of the parametric curve, and we present the *Maximum Balance Criterion*. We also discuss several computational aspects of the numerical algorithm. In section 4, we illustrate the new method by means of numerical examples.

2. GENERAL ASPECTS OF THE MINIMAL PSEUDOINVERSE MATRIX METHOD

The Minimal Pseudoinverse Matriz method is due to [Leonel, A., (1985)], it allows us to find a stable approximation of the pseudoinverse of a matrix and the solution of a system of linear algebraic equations from perturbed data. The main feature of this method is its optimal order of accuracy for consistent and inconsistent systems.

The MPM method is based on the solution of the following extremal problem:

Problem 2.1. Given $A_h \in \Re^{mxn}$ and h > 0, find $\tilde{A}_h$ such that

$$\| \tilde{A}_h^+ \| = \inf \{\|A^+\| : A \in \Re^{mxn}, \|A - A_h\| \le h\}$$

Any solution $\tilde{A}_h$ of this problem is called a *matrix of the MPM method* and $\tilde{A}_h^+$ is called a *minimal pseudoinverse matrix*.

The main results for the MPM Method are the following ([Leonov, A., (1987), (1991)]):

1. Problem 2.1 has a solution for any $A_h \in \Re^{mxn}$ and any $h \ge 0$. The solution need not to be unique.

2. If $\|A_h\| > h$ then all solutions $\tilde{A}_h$ of problem 2.1 satisfy the equality $\| \tilde{A}_h - A_h \| = h$. The following property is related with the estability of the rank of the matrix $\tilde{A}_h$ $(Rg(\tilde{A}_h))$ for sufficiently small h.

3. Let $\tilde{A}_h$ be any solution of problem 2.1 for $0 \le h < h_0(A) = \|A^+\|^{-1}/2$. Then $Rg(\tilde{A}_h) = Rg(A)$.

4. If $0 \le h < h_0(A) = \|A^+\|^{-1}/2$, then

$$\|\tilde{A}_h^+ - A^+\| \le \frac{2h\|A^+\|^2}{(1-2h\|A^+\|^3}.$$

The last of the above results says that $\tilde{A}_h^+$ converges to A^+ when h tends to zero.

To implement the MPM Method it is sufficient to find any solution of problem 2.1. To find a solution, we firstly consider the Singular Value Decomposition (SVD) of A_h [Golub, G., (1989)]:

$$A_h = U_h R_h V_h^t$$

where $R_h = \mathrm{diag}\left(\rho_1^h,\cdots,\rho_M^h\right)$ is a rectangular diagonal matrix with non-negative entries ρ_k^h (k = 1,2,…, M = min(m,n)), which are the singular values of A_h. The matrices U_h and V_h, of size m × n and n × m, respectively, have orthonormal columns. In what follows we describe a procedure to find a solution $\hat{A}_h$ of the problem 2.1 in the form $\hat{A}_h = U_h\hat{D}_h V_h^t$, with $\hat{D}_h$ a rectangular diagonal matrix to be determined.

Consider the following problem:

Problem 2.2. We assume that $h \ge 0$ is given and that the singular values ρ_k^h, k = 1,…,M of A_h are known. Find the number $\hat{\rho}_1 \ge \cdots \ge \hat{\rho}_M$ such that:

$$\sum_{k=1}^{M}\theta(\hat{\rho}_k^2) = \inf\left\{\sum_{k=1}^{M}\theta(\rho_k^2) : \rho_1 \ge \cdots \ge \rho_M \ge 0, \sum_{k=1}^{M}(\rho_k - \rho_k^h)^2 = h^2\right\}$$

$\theta(\rho) = \{\rho^{-1},\ \text{if}\ \ \rho \ne 0;\ \ \text{if}\ \ \rho = 0\}$

The following result shows that a solution of this problem permits us to find a solution $\tilde{A}_h$ of the problem 2.1.

Theorem 2.1. Let $\|A_h\| > h$, and let $\hat{\rho}_1,\cdots,\hat{\rho}_M$ be some solution of problem 2.2. Then a solution of problem 2.1 is given by the matriz

$$\hat{A}_h = U_h\hat{D}_h V_h^t,$$

where $\hat{D}_h = \mathrm{diag}(\hat{\rho}_1,...,\hat{\rho}_M)$ and U_h and V_h are the matrices of the singular value decomposition of A_h.

Now, to find a solution of problem 2.2, Leonov introduces the Lagrange function

$$M^\lambda(\rho_1,\ldots,\rho_M) = \lambda\sum_{k=1}^{M}\theta(\rho_k^2) + \sum_{k=1}^{M}(\rho_k - \rho_k^h)^2 \quad \lambda \ge 0 \quad (2.1)$$

and the following extremal problem is considered:

Problem 2.3. Find $\rho_1(\lambda),\ldots,\rho_M(\lambda)$ such that

$$M^\lambda(\rho_1(\lambda),\ldots,\rho_M(\lambda)) = \inf\{M^\lambda(\rho_1,\ldots,\rho_M) : \rho_1 \ge \ldots \ge \rho_M \ge 0\}$$

This problem has a known solution when λ is fixed. The result is as follows:

Lemma 2.1. [Leonov, A., (1991)] Problem 2.3 is solvable for any $\lambda > 0$. For a given λ, all its solutions are given by

$$\rho_k(\lambda) = \left\{ \rho_k^h x_k(\lambda)\,;\ \text{if } 0 < \lambda < \lambda_k;\ 0, \text{ if } \lambda \geq \lambda_k \right\}$$

$$\rho_k(0) = \rho_k^h,\ k = 1,2,\ldots,M$$

where $\lambda_k = 27\,(\rho_k^h)^4/16$ and $x_k(\lambda)$ is a solution of the equation

$$x^4 - x^3 = \lambda\,(\rho_k^h)^{-4}$$

in the interval [1, 3/2].

The next step is to determine a parameter λ such that the solution of problem 2.3 is the solution of the problem 2.2, and conversely. The parameter λ, except in some special cases, is a solution of the equation

$$\beta(\lambda) = \sum_{k=1}^{M}\left(\rho_k(\lambda) - \rho_k^h\right)^2 = h^2,\ \lambda \geq 0$$

The details of the method can be found in [Leonov, A., (1991)].

3. MAXIMUM BALANCE CRITERION

In the theoretical development of the MPM method, we suppose that the level of perturbation h is known with reliable accuracy. However, this parameter h is frequently unknown or known, but with out accuracy. Consequently, the choice of λ such that solving the equation

$$\beta(\lambda) = h^2$$

has no meaning.

In this section, we introduce a new criterion to choose the parameter λ when h is unknown, which will be called the *Maximum Balance Criterion*. This criterion is based on the selection of a parameter λ in (2.1), which guarantees a good balance between the norm of the minimal pseudoinverse $\tilde{A}_h^+$ and the distance from minimal matrix $\tilde{A}_h$ to A_h, i.e. between the two following contradictory measures

$$\sum_{k=1}^{M}\theta\left(\rho_k^2(\lambda)\right)$$

and

$$\sum_{k=1}^{M}\left(\rho_k(\lambda) - \rho_k^h\right)^2$$

For this selected parameter λ, we can define a level of perturbation $h = \tilde{h}$ such that

$$\tilde{h}^2 = \sum_{k=1}^{M} \left(\rho_k(\lambda) - \rho_k^h \right)^2$$

Then the approximation of the pseudoinverse of A_h given by the *Maximum Balance Criterion* is the minimal pseudoinverse of A_h for $h = \tilde{h}$.

The *Maximum Balance Criterion* is inspired by the L-curve method [Hansen, P.C., (1992), (1993)], which is used to choose the regularization parameter in the solution of ill-conditioned algebraic systems of linear equations when the level of perturbation is unknown [Tikhonov, A.N., Glasko, V.B. (1965)], [Whaba, G., (1977)], [Hansen, P.C. (1992)].

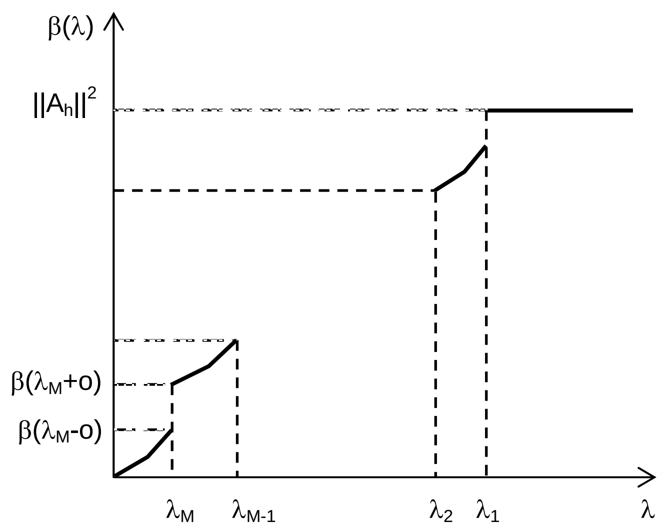

Figure 1. Behaviour of function Beta

What follows, we will study the behaviour of the following functions

$$\beta(\lambda) = \sum_{k=1}^{M} (\rho)$$

with respect to parameter λ. These functions play an important role in the *Maximum Balance Criterion*.

Using the obtained expressions in Lemma 2 for values of $\rho_k(\lambda)$ with $k = 1,\ldots,M$, it directly follows that

$$\beta(0) = 0 \text{ and } \beta(\lambda) = ||A_h||^2 \text{ for } \lambda > \lambda_1$$

The behaviour of $\beta(\lambda)$ for $0 \le \lambda \le \lambda_1$ is studied in [Leonov, A., (1991)]. The graph of $\beta(\lambda)$ is described in the Figure 1.

For the analysis of the behaviour of $\gamma(\lambda)$, we will need the proof of the following theorem.

Theorem 3.1. $\gamma(\lambda)$ is a decreasing monotone function for $0 \le \lambda \le \lambda_1$; it is continuous everywhere except at the points of discontinuity of the first kind $\lambda = \lambda_p$ for $p = 1,\ldots,M$, at which the function is also not well defined. Also,

$$\gamma(0) = ||\, A_h^+ \,||^2 \text{ and } \gamma(\lambda) = 0,\ \lambda > \lambda_1$$

Proof. The function $\gamma(\lambda)$ can be expressed as $\gamma(\lambda) = \xi_1(\lambda) + \ldots + \xi_M(\lambda)$ where the new functions $\xi_k(\lambda)$ are defined as

$$\xi_k(\lambda) = \frac{1}{(\rho_k^h x_k(\lambda))^2} \quad \text{if } 0 < \lambda \le \lambda_k$$

$$\xi_k(\lambda) = 0 \quad \lambda \ge \lambda_k$$

From the definition of $\xi_k(\lambda)$ it is clear that λ_k is a point of discontinuity where

$$\xi_k(\lambda_k - 0) = \frac{1}{(\rho_k^h x_k(\lambda_k))^2},\ \xi_k(\lambda_k + 0) = 0$$

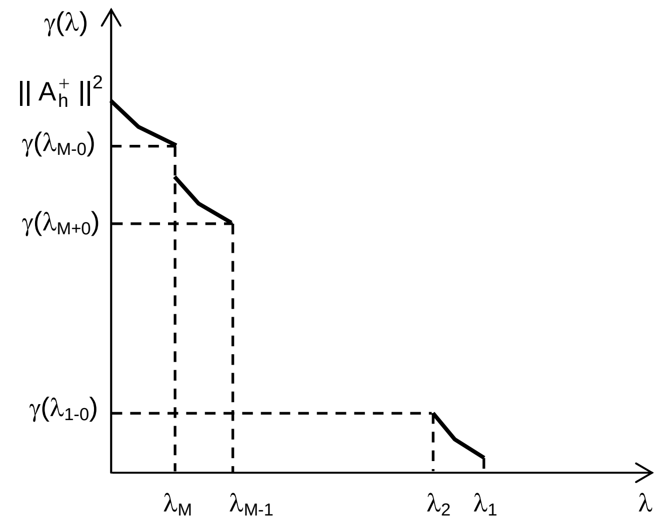

Figure 2. Behaviour of function Gamma

Considering the properties of the function $x_k(\lambda)$ (see Lemma 2.1), it is easy to deduce that $\xi_k(\lambda)$ is continuous and decrease monotonically to a value of λ in $[0,\lambda_k]$. Considering function $\gamma(\lambda)$ as the sum of the function ξ_k, the mentioned properties of $\gamma(\lambda)$ are obvious. The expressions for $\gamma(0)$ and $\gamma(\lambda)$ are obtained straightfoward. The graph of $\gamma(\lambda)$ is as presented in Figure 2. □

The *Maximum Balance Criterion* is based on the analysis of the typical form of the parametric curve $(\beta(\lambda), \gamma(\lambda))$. The main properties of this curve are the following:

- For $\lambda \geq 0$, the curve is not defined outside the intervals $[0,\beta(\lambda_1 + 0)]$ for the abscissas and $[0,\gamma(0)]$ for the ordinates. This property is evident from the graphs 1 and 2.
- For the abscisses, the curve is not defined in the intervals $(\beta(\lambda_k\text{-}0),\ \beta(\lambda_k+0)) = (\beta(\lambda_k\text{-}0),\ \beta(\lambda_k\text{-}0) + 3q(k)(\rho_k^h)^2/4)$ $k = 1,\ldots,M$ where $\beta(\lambda_k+0)$ and $\beta(\lambda_k\text{-}0)$ are the evaluations of β for rigth and left, respectively, and $q(k)$ is the multiplicity of the singular value ρ_k^h of A_h.
- In the intervals $(0,\beta(\lambda_M\text{-}0))$ corresponding to the values of the parameters λ in $(0,\lambda_M)$ and $(\beta(\lambda_k+0),\ \beta(\lambda_{k+1}\text{-}0))$ corresponding to the interval $(\lambda_k, \lambda_{k+1})$, the parametric curve is continuous and monotone decreasing.

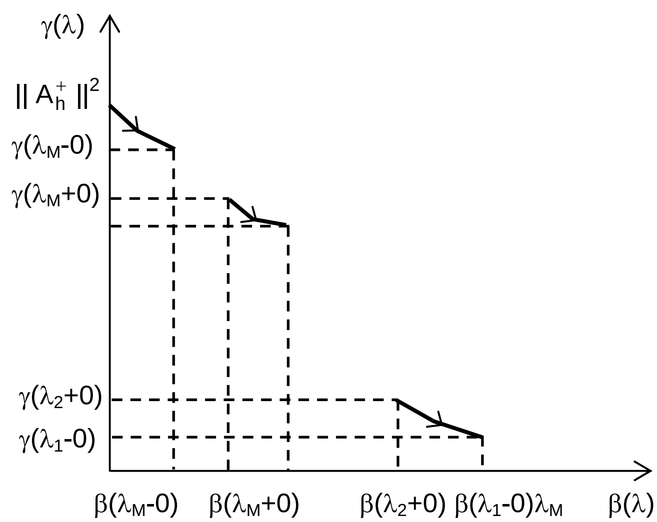

Figure 3. Behaviour of the parametric curve

Then the parametric curve will have the form described in the Figure 3.

Now, we have the tools for presenting our *Maximum Balance Criterion*. First of all, remember that when A_h is an ill-conditioned matrix, in general, we have that the norm

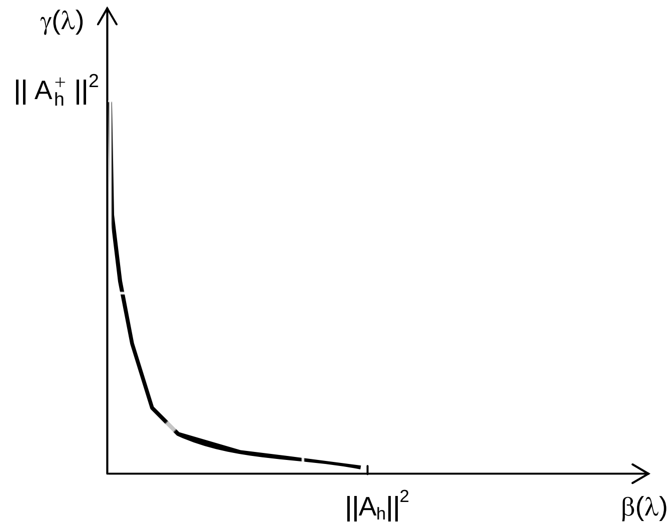

Figure 4. Behaviour of the parametric curve in the case of ill-conditioned matrices

$$\| A_h^+ \| = \sum_{k=1}^{M} \theta(\rho_k^2)$$

is very large with respect to $\|A_h\|^2$. In this case, the plot of the parametric curve $(\beta/\lambda), \gamma(\lambda))$ is plotted in the Figure 4.

If the intervala of indefinition are ignored, it is easy to realize that the parametric curve has an L-shaped appearance. The most horizontal part corresponds to parameters λ that produce matrices $\breve{A}_h$ near to $A_h((\beta/\lambda)$ near to 0), but the norm of the matrix $\breve{A}_h$ is very large. The most vertical segment corresponds to parameters λ such that the associated matrices $\breve{A}_h$ have small norms values but they are quite far from A_h.

From this discussion, it is clear that to have a reasonable choice for parameter λ, it is necessary to obtain a fair balance between β(λ) and γ(λ). In other words, we may choose a parameter λ corresponding to the *corner* of the parametric curve.

The *Maximum Balance Criterion* for choosing the parameter λ has two main parts:

- The construction of the parametric plot γ(λ) vs β(λ).

  The graphical behaviour of the curve reveals considerable information about the degree of ill conditioning of our problem as well as about the variation of the curve depending on parameter λ.

- Computation of the *corner point* of the parametric curve.

  Althought the parametric curve (β/λ), γ(λ)) is easily defined and quite satisfying intuitively, the calculation of the *corner point* has some technical difficulties.

3.1 Numerical issues for locating the *corner point* of the parametric curve

Because our parametric curve has intervals of indefinition, we propose to work with the following set Γ of 3M discrete points:

$(0, \gamma(0)) = (\| A_h^+ \|^2)$

$(\beta(\lambda_p - 0), \gamma(\lambda_p - 0)), (\beta(\lambda_p + 0), \gamma(\lambda_p + 0))$, p = M,…,2

$$\left(\frac{\beta(\lambda_M - 0)}{2}, \frac{\gamma(0) - \gamma(\lambda_M - 0)}{2}\right)$$

$$\left(\frac{\beta(\lambda_0 - 0) - \beta(\lambda_{p+1} + 0)}{2}, \frac{\gamma(\lambda_{p+1} + 0) - \gamma(\lambda_p - 0)}{2}\right), \text{ p = M - 1,…,1}$$

$(\beta(\lambda_1 + 0), \gamma(\lambda_1 + 0)), (||A_h||^2, 0)$

and fit a nice curve through these points.

At first we had an idea based on the use of a cubic spline curve, which has several favorable features in connection with our problem: it is twice differentiable, it can be differentiated in a numerically stable way, and it has local shape-preserving features. As criterion of *corner point* of the parametric curve we choose the point from the set Γ which is the closest to the point of maximum curvature of the cubic spline curve. However, that criterion presented some difficulties. In general, it is quite difficult to find a distribution of knots that allows a good approximation of first and second derivatives of $\gamma(\lambda)$ and $\beta(\lambda)$. Therefore the point of maximum curvature of the spline gives a very sensitive localization of the *corner point* of the parametric curve.

Instead of it we proposed to fit the points of the set Γ by a conic section and to give as the *corner point* of the parametric curve, the one on the discrete curve nearest to the shoulder point of the conic [Hernández, V., (1997)]. More precisely, we consider the triangle T whose vertexes are the points

$b_0 = \left(0, ||A_h^+||^2\right)$, $b_1 = (0,0)$, $b_2 = (||A_h||^2, 0)$

and compute a conic section $c(t)$, $t \in [0,1]$ such that $c(0) = b_0$, $c(1) = b_1$ and $c(t)$ is tangent to the segments $\overline{b_1 b_0}$ and $\overline{b_1 b_2}$ at t = 0 and t = 1, respectively. These conditions define a family of conic sections depending on a free parameter $w_1 > 0$. It can be represented in Bernstein-Bezier form as

$$c(t) = \frac{b_0 B_0^2(t) + w_1 b_1 B_1^2(t) + b_2 B_2^2(t)}{B_0^2(t) + w_1 B_1^2(t) + B_2^2(t)}$$

where

$$B_i^2(t) = \binom{2}{i} t^i (1 - t)^{2-i}, \quad i = 0,1,2$$

are the Bernstein polynomials of degree 2. When $w_1 \to +\infty$ the associated conic section approaches b_1. We use w_1 to select an specific conic from the family that fist the points of the set Γ.

Let w_1^i be the parameter of the unique conic section that satisfied the above conditions and additionally passes through the i-th point of the set Γ. Then, the fitting conic is defined by the parameter

$$w_1^* = \sum_{i=1}^{M} \frac{(w_1^i)}{M}$$

and its shoulder point is

$$s = \frac{b_0 + 2w_1^* b_1 + b_2}{2(1 + w_1)}$$

Hence, to obtain an estimation of the *corner*, we need not compute the derivatives of the fitting curve.

In summary, the algorithm for calculating a pseudoinverse minimal matriz (without any knowledge of the level or perturbation h) based on the *Maximun Balance Criterion* is as follows.

Algorithm MAXBAL:

1.Compute the singular value decomposition of the given matrix $A_h = U_h R_h V_h^t$.

2.Compute the points of discontinuity λ_k, k = 1,…,M of the function $\beta(\lambda)$.

3. Construct the vectors of the described points in the set Γ considered in the parametric curve $(\beta(\lambda), \gamma(\lambda))$.

4.Compute the parameter w_1^* for the fitting conic.

5.Compute the shoulder point s of the fitting conic.

6.Locate the point of the set that Γ that is the closest to the shoulder point of the fitting conic and determine the corresponding value of $\lambda = \lambda(\tilde{h})$.

7.Construct the numbers $\tilde{\rho}_k = \rho_k(\lambda\ (h))$, k = 1,…,M, according to lemma 2.1 and compute the matrix $\tilde{A}_h$ according to theorem 2.1.

8. Determine the minimal pseudoinverse matrix

$$\tilde{A}_h^+ = V_h \tilde{D}_h^+ U_h^t, \qquad \tilde{D}_h = \mathrm{diag}(\tilde{\rho}_1, \cdots, \tilde{\rho}_M)$$

It is important to note that the approximation of the pseudoinverse of A_h given by the *Maximum Balance Criterion* is the minimal pseudoinverse of A_h for $\tilde{h} = \beta(\lambda(\tilde{h}))^{1/2}$.

4. NUMERICAL EXAMPLES

We made the computational implementation of the *Maximum Balance Criterion* proposed in section 3 using MATLAB 4.2 for Windows. In this section, we present our method by means of some numerical examples.

Let A be a matrix of order 30 $\times$ 22. The non-zero singular values of A are:

12.300000000000001
8.500000000000000
3.500000000000001
2.500000000000001
1.700000000000000
0.670000000000000
0.430000000000000
0.100000000000000

It is not hard to obtain a good numerical approximation to the pseudoinverse of A. However, if we applied some perturbation to matrix A then the approximation of the pseudoinverse of A from the perturbed matrix A_h is a very difficult problem, for which the standard techniques are not recommended.

Let's suppose that matrix A has been perturbed by a matrix with random entries of order 10^{-4}. this perturbed matrix A_h is an ill-conditioned matrix with ill-determined numerical rank because there is not a particular gap in the singular value spectrum. The perturbed singular values are

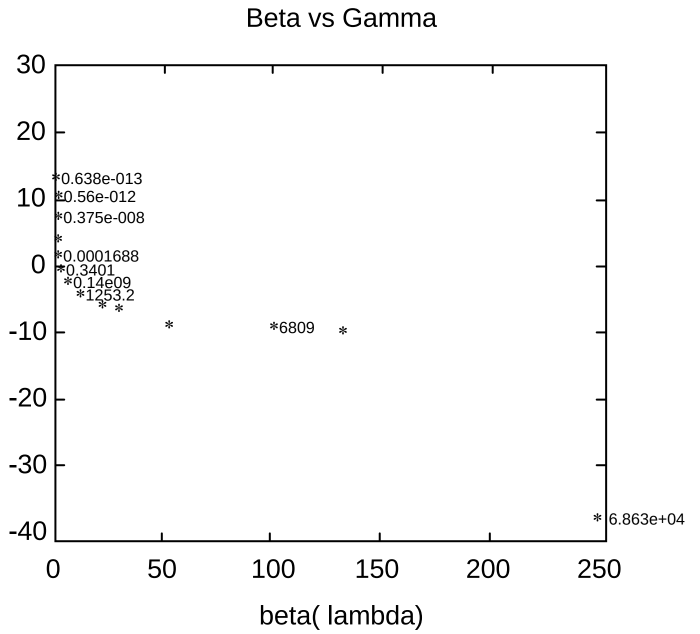

Figure 5.

12.30128711035022	8.49999762751726	3.50001526963961
2.50002785418650	1.70000128699162	0.67000859623878
0.42996955441546	0.09999617159886	0.00023644146929
0.00020735872477	0.00019835676451	0.00017161314264
0.00016306738841	0.00013801322472	0.00012487570343
0.00011680497826	0.00010919290125	0.00010432186159
0.00008129034589	0.00006826773905	0.00004599254587
	0.00004251514023	

In Figure 5, we show some points on the parametric curve $(\beta(\lambda), \gamma(\lambda))$, and their corresponding parameter values λ. Note that the more vertical part corresponds to small values of λ while the more horizontal part is obtained for large values of parameter .

The MAXBAL algorithm locates the *corner point* of the parametric curve for $\lambda = 7.217$. For this value of the parameter, we obtain a minimal pseudoinverse $\tilde{A}_h^+$, in which $\|A^+ - \tilde{A}_h^+\| = 0.4355$.

Table 4.1 shows the norm of the difference between the exact pseudoinverse and the minimal pseudoinverse computed for different values of the parameter h. Note that the results change considerably with respect to h. This shows that the norms of the relative errors of the solution can be seriously affected when the level of perturbation h is not reliably known.

Table 4.1.

1.0
0.5
0.1
0.05
0.00
0.00
0.00
0.00
0.00
0.00

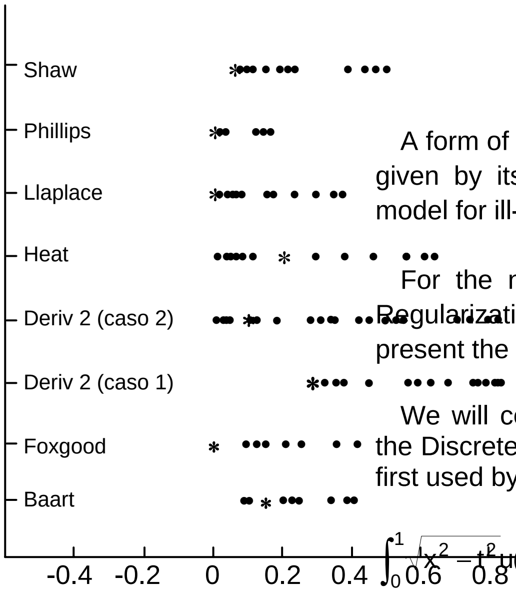

Figure 7: Relative errors in the solution by Max Balance (∗) and MPM method (•)

A form of testing the effectiveness of the *Maximum Balance Criterion* is given by its capacity of reproducing the solution of a discrete linear model for ill-conditioned matrices with ill-determined rank.

For the numerical experimentation, we used the test problems for Regularization Methods given in [Hansen, P.C., (1994)]. Here, we present the results for one of these problems.

We will consider a severely ill-posed problem which does not satisfy the Discrete Picard condition for small singular values. This problem was first used by Fox-Goodwin:

$$\int_0^1 \sqrt{x^2 - t^2}\,u(t)dt = (1/3)\left((1+x^2)^{3/2} - x^3\right)$$ with exact solution u(t) = t.

The integral equation was discretized by a simple cuadrature method for n = 100 (midpoint rule) obtaining the linear system $A_h u = b_\delta$.

Figure 6a shows the exact solution u and the MBC solution $\tilde{u}$, i.e, $\tilde{u} = \tilde{A}_h^+ b_\delta$, where $\tilde{A}_h^+$ is the minimal pseudoinverse given by MBC.

In Figure 6B, we show the different solutions obtained by the minimal method with h known (h = 0.1, 0.01, 0.001, 0.0001) and the exact solution u. Note that the solution given by the minimal method is very dependent on parameter h. The Table 4.2 shown the relative error of the MBC solution and the minimal method with different values of h.

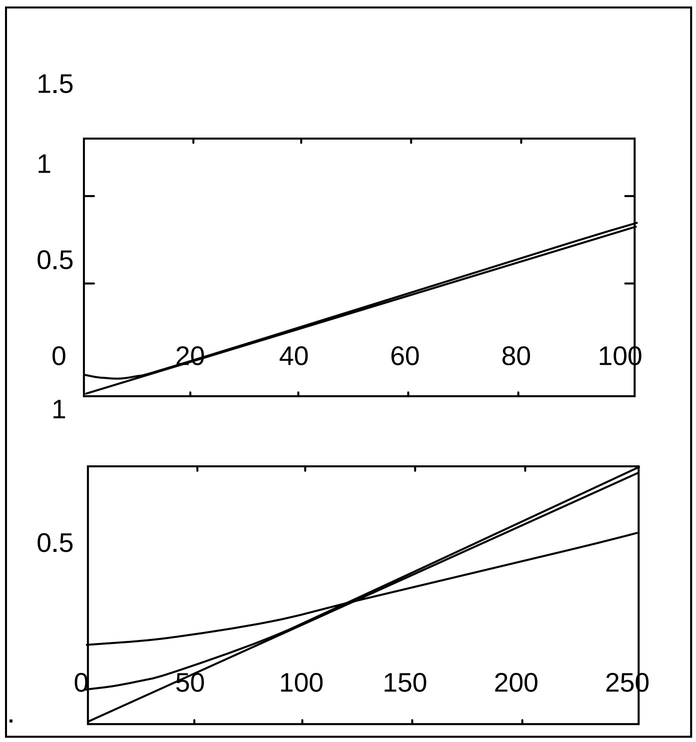

Figure 6.
Above: Exact and MBC solutions.
Below: Minimal solutions for different parameters h

Figure 7 resumes the obtained results for different problems of the collection of the matrices given in [Hansen, P.C. (1994)]. We consider perturbations of order equal to 10^{-3} in the exact matrix A. The symbol '*' represents the relative error of the obtained solution by *Maximum Balance Criterion* applied to the perturbed matrix A_h without the knowledge of the level or perturbation h. The symbol '.' represents the obtained

Table 4.2

Solution x by	**||u - x|| / ||u||**
MBC solution	0.0311
Minimal (h = 0.1)	0.2982
Minimal (h = 0.01)	0.0741
Minimal (h = 0.001)	0.0031
Minimal (h = 0.0001)	9.6e - 04
Minimal (h = 0.00001)	293.340

solutions using the Minimal Method for different values of h in the interval $[10^{-4}, 10^{-1}]$.

In the examples a analysis of the graphic permits to conclude that, when h is unknown, it is better to use Maximum Balance Criterion than Leonov´s Minimal Method.

4.1. CONCLUSIONS

When we want to obtain a good approximation of the pseudoinverse of A, which is known by means of the perturbed matrix A_h and the level of perturbation h is unknown, then the MPM method is not applicable.

Although a rigorous analysis of the *Maximum Balance Criterion* is still lacking, and further research work is needed, we show that if only a rough estimation of h is known or if it is completely unknown, then our criterion is a good alternative, which works reasonably well in practice.

We know it is not possible to construct stable methods for solving ill-posed problems with no use of h [Leonov, A. - Yagola, A., (1995)], [Leonov, A., (1996)]. Nevertheless as the practical problems need to be solved, the strategies based on heuristic approach are important techniques in this undesirable and frequent case.

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